The shell-generated algebra is not an algebra of all compact operators

`MathlibAnnex.CStarAlgebra.CAR.not_isCompactOperatorModel_shellFamilyTarget`

theorem

Excludes an injective representation of the infinite-dimensional unital algebra onto all compact operators.

Statement

For the fixed shell-generated algebra T below and any complete complex Hilbert space K, there is no injective complex-linear star representation $e : T \to B(K)$ whose image is exactly the compact operators on K. The map e is not assumed to preserve the unit.

Assumptions

Let A be the completed CAR algebra with matrix stages $M_{2^n}(\mathbb{C})$ embedded by $a \mapsto a \otimes I_2$. Write f_n for the image of the first diagonal matrix unit, $p_n = f_n - f_{n+1}$, and ω for the root state characterized by $\omega(\iota_n(a)) = a_{00}$ on each stage, where ι_n is its canonical embedding.

Index chosen pure-state Gelfand-Naimark-Segal (GNS) representations by their unitary-equivalence classes $j \in I$, with root index o and representative states φ_j.

A fixed shell family consists of complex-linear unital star automorphisms α_j and elements $w_{j,n} \in A$ satisfying $\varphi_j(\alpha_j(a)) = \omega(a)$, $w_{j,n}^* w_{j,n} = \alpha_j(p_n)$ and $w_{j,n} w_{j,n}^* = p_n$.

At the root, α_o is the identity and $w_{o,n} = p_n$.

Let $\mathcal{H} = \bigoplus_{j \in I} H_j$ be the Hilbert direct sum of these GNS spaces, $\pi = \bigoplus_j \pi_j$ their representation of A, and $b_j = e_j \xi_j \in \mathcal{H}$ its embedded unit cyclic vector, where $\xi_j \in H_j$ is the selected unit cyclic vector and $e_j : H_j \to \mathcal{H}$ is coordinate inclusion.

Choose once a family of unitary operators $L_j \in B(\mathcal{H})$ from the construction of the shell unitaries: $L_j b_j = b_o$, $L_j \pi(\alpha_j(p_n)) = \pi(w_{j,n})$, and $L_o = I_{\mathcal{H}}$. Here $B(\mathcal{H})$ denotes the bounded complex-linear operators on $\mathcal{H}$.

Put $T = C^*(\pi(A), \{L_j : j \in I\}) \subseteq B(\mathcal{H})$, the norm-closed unital star algebra generated by these operators. Let $\iota : A \to T$ be $\iota(a) = \pi(a)$ with its codomain restricted to T, and let $\rho_0 : T \to B(\mathcal{H})$ be inclusion.

The comparison space K may have any dimension and belongs to an arbitrary independent universe. The exact range condition says both that each $e(t)$ is compact and that every compact operator on K equals some $e(t)$. Inner products in the rank-one formula are linear in the second argument.

Conclusion

Thus T is not isomorphic to the C*-algebra of all compact operators on any complete complex Hilbert space K, including the zero Hilbert space. No preservation of the identity is imposed on the candidate representation.

Proof route

Assume such a map exists. Injectivity and $T \neq 0$ first force $K \neq 0$. Since every rank-one operator is in the image, $e(1)$ acts as the identity on every vector: apply $e(1)R = R$ to a rank-one operator R sending a fixed unit vector to the desired vector. Thus $e(1) = I_K$. But $e(1)$ is compact, forcing K to be finite-dimensional. Then $B(K)$ and its complex-linear subspace $e(T)$ are finite-dimensional, contradicting the infinite dimension of T.

Proof steps

1. The zero Hilbert space admits only the zero operator, so it cannot receive an injective representation of the nonzero algebra T.
2. For $K \neq 0$, choose a unit vector z. The rank-one operator $y \mapsto \langle z, y \rangle x$ lies in the image for any x; evaluating $e(1)R = R$ at z gives $e(1)x = x$.
3. A compact identity operator forces finite dimension. Injectivity of the complex-linear map e then transfers finite dimension back to T, a contradiction.

Main citations

- Exact declaration and proof · Exact source
- MathlibAnnex.CStarAlgebra.CAR.RepresentativeShellFamily · Exact source
- MathlibAnnex.CStarAlgebra.CAR.exists_completedAtomicShellModel · Exact source
- MathlibAnnex.CStarAlgebra.CAR.not_finiteDimensional_shellFamilyTarget · Exact source
- MathlibAnnex.CStarAlgebra.CAR.shellFamilySourceHom_injective · Exact source
- MathlibAnnex.CStarAlgebra.CAR.not_finiteDimensional · Exact source
- MathlibAnnex.Analysis.CStarAlgebra.not_isCompactOperatorModel_of_infiniteDimensional · Exact source

Lean source signature (exact)

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theorem not_isCompactOperatorModel_shellFamilyTarget
    (family : RepresentativeShellFamily)
    {K : Type v} [NormedAddCommGroup K] [InnerProductSpace ℂ K]
    [CompleteSpace K]
    (e : ShellFamilyTarget family →⋆ₙₐ[ℂ] (K →L[ℂ] K)) :
    ¬ IsCompactOperatorModel e
```

> Here `ShellFamilyTarget family` is T, and e is the possibly nonunital star representation into $B(K)$. `IsCompactOperatorModel e` combines three conditions: injectivity, compactness of every $e(t)$, and surjectivity onto all compact operators. Its negation excludes an injective star representation of T whose range is exactly the algebra of all compact operators on K.

Lean realization notes

The obstruction uses that T is unital and infinite-dimensional, with infinite dimension established in the target-dimension theorem using its injective copy of the completed CAR algebra. It is distinct from results about the intersection of one represented image with the compact operators.

▼ **Exact Card identity**

Language: en
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Card revision: 1 · SHA-256: 63eb23bc6a8d3e7555fba85b8fa2b3034c6655a355c5a79f27854ee042fbc82b
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