Shell matching fixes the trace of transported flags

`MathlibAnnex.CStarAlgebra.CAR.trace_transportedFlag`

theorem

Recovers exact trace values from the initial and final supports of shell links.

Statement

For the completed CAR algebra with normalized trace τ and a fixed shell family, the transported projection $f_{j,n} = \alpha_j(f_n)$ satisfies $\tau(f_{j,n}) = 2^{-n}$ for every index j and every $n \geq 0$. Here f_n is the root projection at the matrix stage of size 2^n.

Assumptions

Let A be the completed CAR algebra with matrix stages $M_{2^n}(\mathbb{C})$ embedded by $a \mapsto a \otimes I_2$. Write f_n for the image of the first diagonal matrix unit, $p_n = f_n - f_{n+1}$, and ω for the root state characterized by $\omega(\iota_n(a)) = a_{00}$ on each stage, where ι_n is its canonical embedding.

Index chosen pure-state Gelfand-Naimark-Segal (GNS) representations by their unitary-equivalence classes $j \in I$, with root index o and representative states φ_j.

A fixed shell family consists of complex-linear unital star automorphisms α_j and elements $w_{j,n} \in A$ satisfying $\varphi_j(\alpha_j(a)) = \omega(a)$, $w_{j,n}^* w_{j,n} = \alpha_j(p_n)$ and $w_{j,n} w_{j,n}^* = p_n$.

At the root, α_o is the identity and $w_{o,n} = p_n$.

The functional τ is the normalized CAR trace defined in the trace construction. In particular $\tau(f_n) = 2^{-n}$. The same family and the same automorphism α_j are used for all values of n.

Conclusion

Transported flags and root flags have the same trace at each level. This supplies the decaying scalar values needed to estimate their action on vectors implementing the trace.

Proof route

For each shell link, traciality gives $\tau(w_{j,n}^* w_{j,n}) = \tau(w_{j,n} w_{j,n}^*)$. Its support identities turn this into $\tau(f_{j,n}) - \tau(f_{j,n+1}) = \tau(f_n) - \tau(f_{n+1})$. Both flags start at 1, so induction forces equal trace values at every level. The root value is the normalized trace of a rank-one diagonal matrix projection.

Proof steps

1. Use $w_{j,n}^* w_{j,n} = f_{j,n} - f_{j,n+1}$ and $w_{j,n} w_{j,n}^* = f_n - f_{n+1}$ in the trace identity.
2. At $n = 0$ both flags are 1. If their traces agree at n, equality of the successive differences gives agreement at $n + 1$.
3. At stage n, the root projection has exactly one diagonal entry equal to 1. Dividing its matrix trace by 2^n gives the stated value.

Main citations

- Exact declaration and proof · Exact source
- MathlibAnnex.CStarAlgebra.CAR.RepresentativeShellFamily · Exact source
- MathlibAnnex.CStarAlgebra.CAR.RepresentativeShellData · Exact source
- MathlibAnnex.CStarAlgebra.CAR.transportedFlag · Exact source
- MathlibAnnex.CStarAlgebra.CAR.transportedFlag_zero · Exact source
- MathlibAnnex.CStarAlgebra.CAR.representativeLink_initial · Exact source
- MathlibAnnex.CStarAlgebra.CAR.representativeLink_final · Exact source
- MathlibAnnex.CStarAlgebra.CAR.trace_mul_comm · Exact source
- MathlibAnnex.CStarAlgebra.CAR.trace_rootFlag · Exact source
- MathlibAnnex.CStarAlgebra.CAR.trace_transportedFlag_eq_trace_rootFlag · Exact source
- MathlibAnnex.CStarAlgebra.CAR.trace · Exact source

Lean source signature (exact)

```
@[simp]
theorem trace_transportedFlag (family : RepresentativeShellFamily)
    (i : MathlibAnnex.CStarAlgebra.PureState.GNSClass Limit) (n : ℕ) :
    trace (transportedFlag family i n) = (2 ^ n : ℂ)⁻¹
```

> Here `Limit` is A, `trace` is τ, and `rootFlag n` is f_n. The source index `i` is the index j used here. The expression `transportedFlag family i n` is $\alpha_j(f_n) = f_{j,n}$. The right-hand side is the reciprocal of 2^n, regarded as a complex scalar, so the source equality is precisely $\tau(f_{j,n}) = 2^{-n}$.

Lean realization notes

No new automorphism is chosen and trace-invariance of arbitrary automorphisms is not assumed. The proof uses the exact support identities of the given links and finite induction; it does not infer that the projections themselves converge in operator norm.

▼ **Exact Card identity**

Language: en
CARD_CONTENT_COMPLETE
Card UID: lfh:lfh-declaration-card:sha256:6293f0d949a9d46a65ced9dd45fafd88ff8f5e06fc7a4ab38a0a439f8b1feb1e
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Exposition revision: 1 · SHA-256: 85ba692bd237d55892f80c775d498013680ebb72282e8106668d7b36c051df9d
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