

The GNS representation of the target trace extension

MathlibAnnex.CStarAlgebra.CAR.tracialRepresentation

def

The chosen state on the existing target supplies its cyclic GNS representation.

Statement

Let A be the completed CAR algebra and τ its normalized trace. Fix a representative shell family and its chosen unitary links L_j in the selected atomic representation π . Write $T = C^*(\pi(A), \{L_j\})$ for the resulting concrete target and $\iota : A \rightarrow T$, $\iota(a) = \pi(a)$, for its faithful unital source map. This fixes one target and one source map throughout. Let ϕ be the chosen state extension of τ to T . Define $\rho_\phi : T \rightarrow B(H_\phi)$ to be its GNS representation.

Definition

Choose ϕ from `exists_state_extension_trace` and regard that same functional as a positive linear map. Form the GNS completion of T modulo its null vectors, set H_ϕ to that Hilbert space, and let $\xi_\phi = [1]$. The induced left action is

$$\rho_\phi(t)[b] = [tb], \quad \phi(t) = \langle \xi_\phi, \rho_\phi(t)\xi_\phi \rangle.$$

The right-hand side of the definition is exactly the GNS star homomorphism of `traceExtensionPositive` family. This positive-map bundle has the same values as ϕ ; it is not another state. Normalization gives $\|\xi_\phi\| = 1$, so $\xi_\phi \neq 0$ and H_ϕ is nonzero. The GNS construction gives the dense T -orbit. Its restriction realizes τ because $\phi \circ \iota = \tau$. The actual-representation source-cyclicity theorem applies to these two facts and proves the source orbit dense; it is not an assumption smuggled into the definition.

Assumptions

A representative shell family is fixed first. The chosen extension is positive and normalized; its traciality is not needed to perform the GNS construction.

Conclusion

With its canonical unit vector ξ_ϕ , this representation realizes ϕ and has dense target orbit. The separately proved source-cyclicity result also gives

$$\overline{\{\rho_\phi(\iota(a))\xi_\phi : a \in A\}} = H_\phi.$$

Consequently H_ϕ is separable, since A is separable and the displayed orbit map is continuous.

Main citations

- Definition and its exact construction · Exact source
- The fixed concrete target and source map · Exact source

- Faithfulness of the source embedding · Exact source
- Existence used to choose the state · Exact source
- The chosen extension · Exact source
- The positive-map bundle · Exact source
- The target-state GNS Hilbert space · Exact source
- The canonical target-state vector · Exact source
- Normalization of the vector · Exact source
- Nonzero target-state vector · Exact source
- Nontriviality of its Hilbert space · Exact source
- The GNS vector-state identity · Exact source
- The source restriction realizes the trace · Exact source
- Target cyclicity from GNS · Exact source
- Source cyclicity is proved · Exact source
- Separable target-state GNS space · Exact source

Lean source signature (exact)

```
noncomputable def tracialRepresentation (family : RepresentativeShellFamily) :
  Representation (ShellFamilyTarget family) (TracialHilbertSpace family) :=
  (traceExtensionPositive family).gnsStarAlgHom
```

traceExtension is ϕ , and traceExtensionPositive is its positive-map bundle. TracialHilbertSpace and tracialVector denote H_ϕ and ξ_ϕ . The complete defining RHS below := is (traceExtensionPositive family).gnsStarAlgHom. The norm, vector-functional, density and separability statements are separately cited theorems, not omitted fields of this definition.

Lean realization notes

Separability here is a conclusion about the Hilbert space. The algebra T is not assumed norm separable. Injectivity of ρ_ϕ is a separate theorem; irreducibility or purity of ϕ is not asserted.

▼ Exact Card identity

Language: en

CARD_CONTENT_COMPLETE

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