

The completed CAR algebra is infinite-dimensional

MathlibAnnex.CStarAlgebra.CAR.not_finiteDimensional

theorem

Rules out finite dimension by retaining matrix subalgebras of unbounded dimension.

Statement

Let A be the norm completion of the matrix system $A_n = M_{2^n}(\mathbb{C})$ with embeddings $a \mapsto a \otimes I_2$. Then A is not finite-dimensional as a complex vector space.

Assumptions

The algebra A is the fixed completed CAR algebra. For every $n \geq 0$, its canonical stage map $\iota_n : A_n \rightarrow A$ is complex-linear and injective, and $\dim_{\mathbb{C}} A_n = (2^n)^2$. These are established properties of this construction, not new assumptions on an arbitrary algebra.

Conclusion

No finite integer can be the complex dimension of A : every stage contributes an embedded vector space of dimension 4^n , and these dimensions are unbounded.

Proof route

If A had finite dimension k , injectivity of each ι_n would give $(2^n)^2 \leq k$. Take $n = k + 1$. The elementary inequality $k < 2^n \leq (2^n)^2$ contradicts that bound.

Proof steps

1. Assume that A is finite-dimensional and write $k = \dim_{\mathbb{C}} A$.
2. The linear injection ι_{k+1} forces $\dim_{\mathbb{C}} A_{k+1} \leq k$.
3. The stage dimension is $(2^{k+1})^2 > k$, yielding the contradiction.

Main citations

- Exact declaration and proof · Exact source
- MathlibAnnex.CStarAlgebra.CAR.Limit · Exact source
- MathlibAnnex.CStarAlgebra.CAR.ofStage_injective · Exact source
- MathlibAnnex.CStarAlgebra.CAR.finrank_stage · Exact source

Lean source signature (exact)

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theorem not_finiteDimensional : ¬ FiniteDimensional ℂ Limit
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Here `Limit` is the completed CAR algebra $\mathcal{A}$. The expression `FiniteDimensional ℂ Limit` means that $\mathcal{A}$ has finite complex dimension, and the symbol $\neg$ negates that assertion.

Lean realization notes

The conclusion concerns vector-space dimension over $\mathbb{C}$. It does not require an irreducible representation or a claim about compact operators.

▼ Exact Card identity

Language: en

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