An extreme dual functional has a pure-state unitary translate

`MathlibAnnex.Analysis.CStarAlgebra.exists_pureState_unitaryTranslate_of_mem_extremePoints_dual_closedBall`

theorem

Expresses each extreme dual functional as a unitary translate of a pure state.

Statement

Let $A \neq \{0\}$ be a complex C*-algebra, possibly without an identity. Let A^* be the space of bounded complex-linear functionals on A, and let $B_{A^*} = \{\rho : \|\rho\| \leq 1\}$. Extremality is taken for real convex combinations. Write $\widetilde{A} = \mathbb{C} \oplus A$ for the forced C*-unitization and identify A with its canonical ideal in $\widetilde{A}$.

For a unitary $u \in \widetilde{A}$, left multiplication preserves this ideal, and defines the dual translate

$$(T_u\rho)(a) = \rho(ua) \qquad (a \in A).$$

If $\rho \in \mathrm{ext}_{\mathbb{R}}(B_{A^*})$, there are a pure state φ on A and a unitary $u \in \widetilde{A}$ such that

$$T_u\rho = \varphi, \qquad \rho = T_{u^*}\varphi. \tag{1}$$

A pure state is a positive norm-one bounded complex-linear functional that is extreme, for real convexity, in the state space. The two identities in (1) concern the same φ and u.

Assumptions

The nonzero-algebra hypothesis ensures that an extreme element of the dual unit ball has norm one. The source also fixes the compatible C*-order used to express positivity. No identity in A, separability or finite dimension is required. The unitary belongs to the forced unitization even when A happens to have an identity.

Conclusion

The same pure state φ and unitary u satisfy both functional equalities in (1). Positivity, norm one and purity concern φ on the original algebra. The assertion does not require or assert uniqueness of the unitary.

Main citations

- Exact decomposition declaration: `exists_pureState_unitaryTranslate_of_mem_extremePoints_dual_closedBall`.

- Ryotaro Tanaka, arXiv:2504.05551v1, §5: the paper's facial and pure-state comparison. The forced-unitization and original-algebra coefficient construction here follow the pinned source.

Lean source signature (exact)

The declaration prefix below is a contiguous source excerpt, ending before its proof assignment.

```
theorem exists_pureState_unitaryTranslate_of_mem_extremePoints_dual_closedBall
    {rho : StrongDual ℂ A}
    (hrho : rho ∈ (closedBall (0 : StrongDual ℂ A) 1).extremePoints ℝ) :
    ∃ phi : StrongDual ℂ A, IsPureNonUnitalState A phi ∧
      ∃ u : unitary (Unitization ℂ A),
        unitaryTranslate u rho = phi ∧ rho = unitaryTranslate (star u) phi
```

In the source	Mathematical meaning
Exact declaration `{A : Type u}, [NonUnitalCStarAlgebra A], [Nontrivial A]`	The nonzero complex C*-algebra. An identity is not assumed.
Exact declaration `[PartialOrder A], [StarOrderedRing A]`	The compatible C*-order needed for positive functionals and states.
Exact declaration `rho : StrongDual ℂ A`	The bounded complex-linear functional ρ on the original algebra.
Exact declaration `hrho : rho ∈ (closedBall ... 1).extremePoints ℝ`	Extremality in the entire norm-closed dual unit ball, with real convexity. It implies norm one here.
Exact declaration `∃ phi : StrongDual ℂ A, ...`	One bounded complex-linear functional φ is selected in the same dual space as ρ. Positivity is supplied by the following predicate, not by this type alone.
Exact declaration `IsPureNonUnitalState A phi`	Extremality among positive norm-one functionals, including membership in that state space. It supplies positivity and norm one for the same φ.
Exact declaration `∃ u : unitary (Unitization ℂ A), ...`	One unitary in the forced unitization is selected after the pure state; it need not be an element of A.
Exact declaration `unitaryTranslate u rho = phi`	The equality of the whole functionals $T_u\rho = \varphi$, hence equality at every $a \in A$.
The equality with **Exact declaration** `star u`	The whole inverse-translation equality $\rho = T_{u^*}\varphi$, using the same φ and u in both equalities.

Exact surrounding source context (separate excerpts)

Source lines 35–35:

```
variable {A : Type u} [NonUnitalCStarAlgebra A] [Nontrivial A]
```

Source lines 56–56:

```
variable [PartialOrder A] [StarOrderedRing A]
```

Exact declaration for the source guide

```
theorem exists_pureState_unitaryTranslate_of_mem_extremePoints_dual_closedBall
    {rho : StrongDual ℂ A}
    (hrho : rho ∈ (closedBall (0 : StrongDual ℂ A) 1).extremePoints ℝ) :
    ∃ phi : StrongDual ℂ A, IsPureNonUnitalState A phi ∧
      ∃ u : unitary (Unitization ℂ A),
        unitaryTranslate u rho = phi ∧ rho = unitaryTranslate (star u) phi := by
```