Faithfulness of a representative of the unique irreducible-representation class

`MathlibAnnex.Analysis.CStarAlgebra.NonUnitalCStarRepresentation.injective_of_singleton`

Shows that a representative of the unique irreducible-representation class of a nonzero C*-algebra is faithful.

Statement

Let A be a nonzero complex C*-algebra, not assumed unital, H a complex Hilbert space, and π a representation of A on H that is a representative of the unique unitary-equivalence class of nonzero irreducible representations of A. No separability assumption on H is required. Then π is injective.

Assumptions

Let A be a nonzero complex C*-algebra, not assumed unital, H a complex Hilbert space, and π a representation of A on H that is a representative of the unique unitary-equivalence class of nonzero irreducible representations of A. No separability assumption on H is required.

Conclusion

Then π is injective.

Proof idea

For $\pi(x-y)=0$, the source assumes $x-y\neq 0$, embeds it into the minimal unitization, and finds a pure state detecting its star-square. Its GNS representation restricts to a nonzero irreducible representation of A. The unique-class hypothesis gives a unitary equivalence to π, which would force the detected element to act as zero there, a contradiction.

Proof steps

1. Exact Lean statement: ∀ {A : Type u} [inst : NonUnitalCStarAlgebra A] [inst_1 : PartialOrder A] [StarOrderedRing A] {H : Type v} [inst_3 : NormedAddCommGroup H] [inst_4 : InnerProductSpace ℂ H] [inst_5 : CompleteSpace H] [Nontrivial A] (pi : MathlibAnnex.Analysis.CStarAlgebra.NonUnitalCStarRepresentation A H), pi.IsSingletonIrreducibleModel → Function.Injective ⇑pi

2. IsSingletonIrreducibleModel.{u,v,u} π → Injective π.

3. For $\pi(x-y)=0$, the source assumes $x-y\neq 0$, embeds it into the minimal unitization, and finds a pure state detecting its star-square. Its GNS representation restricts to a nonzero irreducible representation of A. The unique-class hypothesis gives a unitary equivalence to π, which would force the detected element to act as zero there, a contradiction.

Main citations

MathlibAnnex.Analysis.CStarAlgebra.NonUnitalCStarRepresentation.IsSingletonIrreducibleModel

Exact formal dependency; inspect the linked Card and exact source.

MathlibAnnex.Analysis.CStarAlgebra.exists_pureState_nonzero_on_star_mul_self

Exact formal dependency; inspect the linked Card and exact source.

MathlibAnnex.Analysis.CStarAlgebra.isIrreducible_pureState_gnsStarAlgHom

Exact formal dependency; inspect the linked Card and exact source.

Lean source signature (exact)

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theorem injective_of_singleton [Nontrivial A]
(pi : NonUnitalCStarRepresentation A H)
(hsingle : IsSingletonIrreducibleModel.{u, v, u} pi) :
Function.Injective pi
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Read exact MathlibAnnex v0.4.0 source

Exact Card identity

Stable Card ID: 66a6654fa9d1e8607b27f931148203017898635ae004e222a22fd8c88c82d2f2

Card SHA-256: 3753106ba648e2a30ddde2adad74dbe93df42d065d9a8b9d3790136458806884

Approved exposition SHA-256: feea6dc38ab8fb94efd870fe5ea3e538156ecce877affe0f70e8b7382ae72a1c

AI-assisted material. Selected, edited, and reviewed by Ryotaro Tanaka. The recorded review is source–exposition correspondence, not an independent proof audit.