The atomic direct sum of the selected CAR representations

`MathlibAnnex.CStarAlgebra.CAR.selectedAtomicRepresentation`

def

All selected pure-GNS fibers of the completed CAR algebra act together on one Hilbert direct sum.

Statement

Let A be the completed CAR algebra, the norm completion of the unital matrix system $M_{2^n}(\mathbb{C})$ with embeddings $a \mapsto a \otimes I_2$. Let ω be its distinguished pure product-vector state. Index pure-state GNS equivalence classes by I, choose one representative φ_j in each class with $\varphi_o = \omega$ at $o = [\omega]$, and denote its GNS data by (H_j, π_j, ξ_j). Write $\mathcal{H} = \bigoplus_{j \in I} H_j$ for the Hilbert direct sum, $e_j : H_j \to \mathcal{H}$ for coordinate inclusion, and $\pi(a)(x_j)_j = (\pi_j(a)x_j)_j$. Here the sum consists of square-summable families and has no countability restriction on I. The selected atomic representation is this coordinatewise unital representation $\pi : A \to B(\mathcal{H})$.

Definition

The common estimate $\|\pi_j(a)\| \leq \|a\|$ bounds every coordinate action uniformly in j. Hence the coordinatewise formula defines a bounded operator on the square-summable families, with $\|\pi(a)\| \leq \|a\|$. Multiplication, the unit and complex linearity hold coordinatewise. The inner product on the direct sum shows that taking adjoints also holds coordinatewise, giving $\pi(a^*) = \pi(a)^*$. The existing arbitrary-index atomic construction packages these properties.

Assumptions

The completed CAR algebra and its root pure state are fixed. Every fiber is the complete GNS Hilbert space of the chosen state. The index set need not be countable, and the Hilbert direct sum is not assumed separable.

Conclusion

For every $a \in A$, $x \in \mathcal{H}$ and $j \in I$, the defining formula is $(\pi(a)x)_j = \pi_j(a)x_j$. In particular, $\pi(a)e_j x = e_j \pi_j(a)x$ for $x \in H_j$. Thus the selected individual actions are the coordinate restrictions of one displayed representation.

Main citations

- Definition and its exact construction · Exact source
- The selected individual GNS action · Exact source
- The Hilbert direct sum of selected fibers · Exact source
- Coordinate inclusion into that Hilbert direct sum · Exact source
- Uniformly bounded coordinatewise operator · Exact source
- The arbitrary-index star representation · Exact source
- Coordinate formula for the atomic action · Exact source
- Separate faithfulness of the selected CAR sum · Exact source

Lean source signature (exact)

```
noncomputable def selectedAtomicRepresentation :
    Representation Limit
      (MathlibAnnex.CStarAlgebra.PureState.SelectedAtomicHilbert completedRootPureState) :=
  atomicRepresentation
    (MathlibAnnex.CStarAlgebra.PureState.selectedRepresentation completedRootPureState)
```

> Here `Limit` is A and `completedRootPureState` is ω bundled with its purity proof. `PureState.selectedRepresentation completedRootPureState` is the family $(\pi_j)_{j\in I}$. The RHS `atomicRepresentation` assembles this family into π on `SelectedAtomicHilbert`, which is $\mathcal{H}$. Coordinate inclusions e_j are the separately defined `selectedEmbedding`; they are used to state the coordinate restriction formula.

Lean realization notes

The symbol π_j denotes one fiber action and π denotes their direct sum. The cited faithfulness theorem is a separate property of this CAR representation, obtained from its faithful root summand; faithfulness is not a new input or a field added to this definition.

▼ **Exact Card identity**

Language: en
CARD_CONTENT_COMPLETE
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