Assembling selected GNS cyclic subspaces into an isometric source representation

`MathlibAnnex.CStarAlgebra.CAR.exists_selectedAtomicCyclicIsometry`

theorem

Joins mutually orthogonal pure-state cyclic copies and controls the limiting flag projections on their sum.

Statement

For the shell family and selected GNS spaces below, let $\sigma : A \to B(K)$ be a unital complex star representation on a complete complex Hilbert space. Suppose unit vectors $\eta_j \in K$ satisfy $\langle \eta_j, \sigma(a)\eta_j \rangle = \varphi_j(a)$ for every $j \in I$ and $a \in A$. Put $M_j = \overline{\{\sigma(a)\eta_j : a \in A\}}$ and let P_j be the orthogonal projection onto $\bigcap_n \operatorname{ran} \sigma(f_{j,n})$.

There is a complex-linear isometry $W : \mathcal{H} \to K$ such that $W(e_j H_j) \subseteq M_j$, $W(e_j \xi_j) = \eta_j$ and $P_j W x \in \mathbb{C}\eta_j$ for every j and $x \in \mathcal{H}$. For every $a \in A$, it satisfies $W\pi(a) = \sigma(a)W$.

Assumptions

Let A be the completed CAR algebra, the completion of the matrix stages $M_{2^n}(\mathbb{C})$ under $a \mapsto a \otimes I_2$. Let f_n be the image of the first diagonal matrix unit, so $f_0 = 1$ and f_n decreases through projections. The root state ω takes the value a_{00} on each matrix stage.

Choose one pure state φ_j from each unitary-equivalence class $j \in I$ of pure-state Gelfand-Naimark-Segal (GNS) representations, with root index o and $\varphi_o = \omega$. Write (H_j, π_j, ξ_j) for its complete complex GNS space, unital star representation and unit cyclic vector. Set $\mathcal{H} = \bigoplus_{j \in I} H_j$, $\pi = \bigoplus_j \pi_j$, and let $e_j : H_j \to \mathcal{H}$ be the coordinate embedding. No countability of I is assumed. Inner products are linear in the second argument.

A supplied shell family consists of unital complex star automorphisms α_j and elements $w_{j,n} \in A$. Put $f_{j,n} = \alpha_j(f_n)$. The identities are $\varphi_j \circ \alpha_j = \omega$, $w_{j,n}^* w_{j,n} = f_{j,n} - f_{j,n+1}$ and $w_{j,n} w_{j,n}^* = f_n - f_{n+1}$. At the root, α_o is the identity and $w_{o,n} = f_n - f_{n+1}$. The family is an input, not an existence conclusion. The vectors η_j are given simultaneously, one for each chosen class. No irreducibility of σ and no density of any individual orbit in all of K is assumed. The algebra and the representations are unital, so each η_j belongs to M_j.

Conclusion

The isometry identifies the atomic Hilbert sum with a closed subspace reducing σ. Each ambient limiting flag projection sends this range into the indicated cyclic line. This is control on the isometry range; it is not a claim that the entire ambient fixed space has dimension one.

Proof route

Identify each cyclic restriction with its selected GNS representation by pointed cyclic transport. Orthogonality of inequivalent cyclic pieces allows these identifications to extend to the arbitrary-index Hilbert sum. The own-piece compression identity and vanishing on other cyclic pieces give the projection assertion; continuity passes the source intertwining law through the convergent sum.

Proof steps

1. On M_j, the restricted representation has the unit cyclic vector η_j with pure state φ_j, and is irreducible. Equality of vector states gives a unitary $U_j : H_j \to M_j$ sending ξ_j to η_j and intertwining π_j with the restriction of σ. It extends the orbit map $\pi_j(a)\xi_j \mapsto \sigma(a)\eta_j$.

2. For $i \neq j$, the orthogonal projection from M_j to M_i intertwines their restrictions, since the cyclic subspaces reduce σ. A nonzero such intertwiner between irreducible representations would give a unitary equivalence by Schur's lemma, contradicting the distinct selected GNS classes. Thus $M_i \perp M_j$. The maps U_j, followed by inclusion into K, form an orthogonal family of isometries.

3. For a fixed i, the state identity gives $\varphi_i(f_{i,n}) = 1$, hence $\sigma(f_{i,n})\eta_i = \eta_i$. Norm compression implies $P_i\sigma(a)P_i = \varphi_i(a)P_i$, so $P_i\sigma(a)\eta_i = \varphi_i(a)\eta_i$. Density of the orbit and closedness of the line give $P_iM_i \subseteq \mathbb{C}\eta_i$. If $x \in M_j$ with $j \neq i$ and $P_ix \neq 0$, normalizing P_ix yields a vector with state φ_i. Its cyclic space is orthogonal to M_j. Then $\langle P_ix, x\rangle = \|P_ix\|^2 = 0$, a contradiction. Hence $P_iM_j = 0$.

4. For $x = (x_j) \in \mathcal{H}$, define $Wx = \sum_j U_jx_j$, with each summand viewed in K. Orthogonality makes this an unconditionally norm-convergent Hilbert sum and gives $\|Wx\|^2 = \sum_j \|x_j\|^2$. The coordinate formulas give $W(e_j\xi_j) = \eta_j$. Applying the bounded operator P_i kills every summand except the i-th, whose image lies in $\mathbb{C}\eta_i$. Applying the bounded operator $\sigma(a)$ termwise gives $W\pi(a) = \sigma(a)W$.

Main citations

- Exact declaration and proof · Exact source
- MathlibAnnex.CStarAlgebra.CAR.RepresentativeShellFamily · Exact source
- MathlibAnnex.CStarAlgebra.CAR.selectedAtomicRepresentation · Exact source
- MathlibAnnex.CStarAlgebra.CAR.exists_selectedCyclicUnitary · Exact source
- MathlibAnnex.CStarAlgebra.CAR.isOrtho_cyclicSubspace_of_selectedStates · Exact source
- MathlibAnnex.CStarAlgebra.CAR.initialFixedProjection_maps_ownCyclic · Exact source
- MathlibAnnex.CStarAlgebra.CAR.initialFixedProjection_eq_zero_on_otherCyclic · Exact source
- MathlibAnnex.CStarAlgebra.CAR.exists_surjective_selectedAtomicCyclicIsometry · Exact source
- MathlibAnnex.CStarAlgebra.CAR.completedRootPureState · Exact source

Lean source signature (exact)

```
theorem exists_selectedAtomicCyclicIsometry
    (family : RepresentativeShellFamily)
    {K : Type v} [NormedAddCommGroup K] [InnerProductSpace ℂ K]
    [CompleteSpace K] (sigma : Representation Limit K)
    (eta : MathlibAnnex.CStarAlgebra.PureState.GNSClass Limit → K)
    (heta : ∀ i, ‖eta i‖ = 1)
    (hstate : ∀ i, Representation.vectorFunctional sigma (eta i) =
      (MathlibAnnex.CStarAlgebra.PureState.representative completedRootPureState i).1) :
    ∃ W : MathlibAnnex.CStarAlgebra.PureState.SelectedAtomicHilbert completedRootPureState →ₗᵢ[ℂ] K,
      (∀ i x, W (MathlibAnnex.CStarAlgebra.PureState.selectedEmbedding completedRootPureState i x) ∈
        Representation.cyclicSubspace sigma (eta i)) ∧
      (∀ i, W (MathlibAnnex.CStarAlgebra.PureState.selectedEmbedding completedRootPureState i
        (MathlibAnnex.CStarAlgebra.PureState.selectedVector completedRootPureState i)) = eta i) ∧
      (∀ i x, commonFixedProjection
        (fun n ↦ sigma (transportedFlag family i n)) (W x) ∈
          ℂ ∙ eta i) ∧
      ∀ a,
        W.toContinuousLinearMap.comp
            (selectedAtomicRepresentation a) =
          (sigma a).comp
            W.toContinuousLinearMap
```

> Here `Limit` is A, `completedRootPureState` is the root state ω bundled with its purity proof, and `SelectedAtomicHilbert completedRootPureState` is $\mathcal{H}$. `selectedEmbedding ... i` is e_j, `selectedVector ... i` is ξ_j, and `eta i` is η_j, with source index `i` corresponding to the mathematical index j. `cyclicSubspace sigma (eta i)` is M_j and `commonFixedProjection` is P_j. The arrow `→ₗᵢ[ℂ]` asserts a complex-linear isometry; it does not assert surjectivity. `W.toContinuousLinearMap` is the same map used in the operator identity.

Lean realization notes

Surjectivity is not part of this theorem. The surjective cyclic-sum theorem proves it in the generated-target setting with additional hypotheses. The present hypotheses already supply the vector states; they do not assert that every representation of the CAR algebra contains all selected classes.

▼ **Exact Card identity**

Language: en
CARD_CONTENT_COMPLETE
Card UID: lfh:lfh-declaration-card:sha256:67cdb16a93cf0accba9a49b2a23802225c8252938eb49ca5542397f7374dbc77
Card revision: 1 · SHA-256: 5b9f5d0a4cfb13095b600fce83d7654403ab06f9f73800276dece4ac88a2ba89
Exposition revision: 1 · SHA-256: 9a2fd2ad5c81a0200c831f362bc88b3039ffbaa404aa904a04216ef9baeaf083
Source commit: 437e6e46228bbb8e91211ebded349d7a30020e73
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