An irreducible CAR representation has no nonzero compact image

`MathlibAnnex.CStarAlgebra.CAR.eq_zero_of_isCompactOperator_image`

theorem

Simplicity and infinite dimensionality rule out compact operators coming from nonzero CAR elements.

Statement

Let A be the completed CAR algebra: the norm completion of the increasing union of $A_n = M_{2^n}(\mathbb{C})$ under the unital embeddings $a \mapsto a \otimes I_2$ (with the fixed coordinate reindexing). Write $\iota_n : A_n \to A$ for the canonical isometric inclusion and $e_{ij}^{(n)} = \iota_n(E_{ij})$ for its matrix units. A representation is a unital complex star homomorphism into the bounded operators on a complex Hilbert space. Irreducibility means nonzero action and no proper nonzero closed reducing subspace. Let $\pi : A \to B(H)$ be irreducible on a complex Hilbert space H. If $a \in A$ and $\pi(a)$ is compact, then $a = 0$. In particular $\pi(A) \cap \mathcal{K}(H) = \{0\}$.

Assumptions

The representation is unital and irreducible in the nonzero, closed-reducing-subspace sense. The space H is complete. Compactness is assumed only for the represented operator $\pi(a)$.

Conclusion

No nonzero element of A can act compactly in this representation. Faithfulness makes the vanishing conclusion a statement about a itself, not merely about its image.

Proof route

Simplicity makes the nonzero unital representation faithful. The preimage of the compact operators is a closed two-sided ideal of A. If it were all of A, the identity on H would be compact; this would make H, and then the faithfully represented algebra A, finite-dimensional. That contradicts infinite dimensionality of the CAR limit.

Proof steps

1. The exact result `representation_injective` gives injectivity of π from simplicity and the nonzero irreducible action.
2. Let $J = \{x \in A : \pi(x) \text{ is compact}\}$. Compact operators form a norm-closed two-sided ideal in $B(H)$, and the representation is continuous, so J is a norm-closed two-sided ideal in A.
3. Simplicity yields $J = \{0\}$ or $J = A$. In the second case $\pi(1) = I_H$ is compact, so H is finite-dimensional. Injectivity then embeds A linearly in the finite-dimensional space $B(H)$, a contradiction. Therefore $J = \{0\}$ and $a = 0$.

Main citations

- The stated existence or structural result · Exact source
- Faithfulness of an irreducible CAR representation · Exact source

- Simplicity of the CAR limit · Exact source
- Infinite dimensionality of the CAR limit · Exact source
- Closed compact-preimage ideal and finite-dimensional contradiction · Exact source
- Representation and nonzero irreducibility conventions · Exact source

Lean source signature (exact)

```
theorem eq_zero_of_isCompactOperator_image
    (rho : Representation Limit H) (hrho : rho.IsIrreducible)
    {a : Limit} (ha : IsCompactOperator (rho a)) : a = 0
```

> Here `rho` is π, `rho.IsIrreducible` includes nonzero action, and `IsCompactOperator (rho a)` is the compactness of $\pi(a)$. Operators `H →L[ℂ] H` are bounded complex-linear maps. The source concludes `a = 0`; it does not only conclude that `rho a` vanishes.

Lean realization notes

This does not say that $B(H)$ has no compact operators. It describes the intersection of its compact ideal with the represented CAR algebra. Infinite dimensionality of A is a proved property of the fixed CAR limit, not an extra assumption on an arbitrary algebra.

▼ **Exact Card identity**

Language: en

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