Pure-state GNS irreducibility

`MathlibAnnex.Analysis.CStarAlgebra.isIrreducible_pureState_gnsStarAlgHom`

Turns purity of a state into irreducibility of its canonical GNS representation.

Statement

Let A be a unital complex C*-algebra and let φ be a pure state of A. The canonical GNS star-algebra homomorphism built from the positive map associated to φ is irreducible.

Assumptions

Let A be a unital complex C*-algebra and let φ be a pure state of A.

Conclusion

The canonical GNS star-algebra homomorphism built from the positive map associated to φ is irreducible.

Proof idea

The source invokes the cyclic-vector irreducibility criterion: the GNS cyclic vector has norm one and dense orbit, its vector functional equals φ by the GNS inner-product identity, and purity supplies the criterion's extremality premise.

Proof steps

1. Exact Lean statement: ∀ {A : Type u} [inst : CStarAlgebra A] [inst_1 : PartialOrder A] [inst_2 : StarOrderedRing A] (phi : A →L[ℂ] ℂ) (hphi : phi ∈ MathlibAnnex.Analysis.CStarAlgebra.stateSpace A), MathlibAnnex.Analysis.CStarAlgebra.IsPureState A phi → (MathlibAnnex.Analysis.CStarAlgebra.positiveLinearMapOfMemStateSpace phi hphi).gnsStarAlgHom.IsIrreducible

2. IsIrreducible ((positiveLinearMapOfMemStateSpace φ hφ).gnsStarAlgHom).

3. The source invokes the cyclic-vector irreducibility criterion: the GNS cyclic vector has norm one and dense orbit, its vector functional equals φ by the GNS inner-product identity, and purity supplies the criterion's extremality premise.

Main citations

MathlibAnnex.Analysis.CStarAlgebra.IsPureState

Exact formal dependency; inspect the linked Card and exact source.

Lean source signature (exact)

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theorem isIrreducible_pureState_gnsStarAlgHom
(phi : A →L[ℂ] ℂ) (hphi : phi ∈ stateSpace A) (hpure : IsPureState A phi) :
StarAlgHom.IsIrreducible
(positiveLinearMapOfMemStateSpace phi hphi).gnsStarAlgHom
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Read exact MathlibAnnex v0.4.0 source

Exact Card identity

Stable Card ID: 68395281a7478e8c956a7482013ed5dd68c4d3f2107f38cf4f8e9117a8a99a4a

Card SHA-256: 54d38276b2624f4bbb831bdcdc75bbba5ee6d940ee1923e2674d469c88464230

Approved exposition SHA-256: 40706299e81e72561e0a9ebb2439c8f4f6823ee47507f806c18df8cbf869788a

AI-assisted material. Selected, edited, and reviewed by Ryotaro Tanaka. The recorded review is source–exposition correspondence, not an independent proof audit.