Lifting a corner involution to an ambient CAR unitary

`MathlibAnnex.CStarAlgebra.CAR.exists_liftedCornerExponential_apply_eq_involution`

theorem

A prescribed unitary involution on a represented root corner can be realized on finitely many vectors by one lifted corner exponential.

Statement

Let A be the completed CAR algebra: the norm completion of the increasing union of $A_n = M_{2^n}(\mathbb{C})$ under the unital embeddings $a \mapsto a \otimes I_2$ (with the fixed coordinate reindexing). Write $\iota_n : A_n \to A$ for the canonical isometric inclusion and $e_{ij}^{(n)} = \iota_n(E_{ij})$ for its matrix units. Let H be a nonzero complex Hilbert space and $\rho : A \to B(H)$ an irreducible unital star representation. Fix a stage n, put $e = e_{00}^{(n)}$, and let $R = \operatorname{ran} \rho(e)$. Suppose $U : R \to R$ is a surjective complex-linear isometry with $U^2 = I_R$. For a finite family $v_0, \ldots, v_{d-1} \in R$, there is $h = h^* \in A$, supported by e on both sides, such that the unitary $u = \sum_i e_{i0}^{(n)} \big(e \exp(ih)\big) e_{0i}^{(n)}$ satisfies $\rho(u)v_j = Uv_j$ for every j.

Assumptions

The Hilbert space is complete and nonzero; irreducibility here means that the representation has no proper nonzero closed reducing subspace. The integer d may be zero. The selected vectors need not span R. The involution is defined on the whole corner Hilbert space and is unitary there.

Conclusion

The same supported self-adjoint generator gives the ambient unitary and all the prescribed finite actions. Support means $eh = h = he$. The sum amplifies the corner unitary $e \exp(ih)$ to a unitary of the unital algebra A.

Proof route

The involution determines an orthogonal projection $Q = (I_R - U)/2$. Let T be the zero extension of πQ from R to H, where π denotes the real circle constant. On the finite span of the selected vectors and their images under U, $\exp(iT)$ acts as U. Supported self-adjoint interpolation realizes this exponential by a corner element; matrix amplification preserves its action on root-corner vectors.

Proof steps

1. Since U is a unitary involution, $Q = (I_R - U)/2$ is an orthogonal projection. Let T be the zero extension of πQ to H. It is self-adjoint, has range in R, and leaves the finite-dimensional space $E = \operatorname{span}\{v_j, Uv_j\}$ invariant. On R, $\exp(i\pi Q) = I_R - 2Q = U$.

2. The supported exponential interpolation result gives a self-adjoint $h \in eAe$ whose corner exponential acts as $\exp(iT)$ on E. Its proof first interpolates T on E with $\|h\| \leq 2\|T\|$, then uses invariance of E to compare the exponentials there.

3. For $z = e\exp(ih)$, the matrix-unit amplification $L_n(z) = \sum_i e^{(n)}_{i0} z e^{(n)}_{0i}$ is unitary. On a root-corner vector its represented action reduces to that of z. Hence $\rho(L_n(z))v_j = Uv_j$. The route explanation below records the accompanying stage-central paths separately.

Main citations

- The stated existence or structural result · Exact source
- The represented root-corner subspace · Exact source
- Corner exponential realizes the involution on the family · Exact source
- Ambient amplification preserves root-corner action · Exact source
- Supported exact exponential interpolation · Exact source
- The amplified corner exponential · Exact source
- Representation and nonzero irreducibility conventions · Exact source
- Equivalence of the two irreducibility interfaces on a nonzero Hilbert space · Exact source

Lean source signature (exact)

```
theorem exists_liftedCornerExponential_apply_eq_involution
    {H : Type*} [NormedAddCommGroup H] [InnerProductSpace ℂ H]
    [CompleteSpace H] [Nontrivial H]
    (rho : Representation Limit H) (hrho : StarAlgHom.IsIrreducible rho)
    (n d : ℕ) (v : Fin d → rootCornerSubspace rho n)
    (U : rootCornerSubspace rho n ≃ₗᵢ[ℂ] rootCornerSubspace rho n)
    (hU : ∀ x, U (U x) = x) :
    ∃ (h : selfAdjoint Limit)
      (heh : limitMatrixUnit n 0 0 * (h : Limit) = h)
      (hhe : (h : Limit) * limitMatrixUnit n 0 0 = h),
      ∀ i, rho (liftedCornerExponential n h heh hhe : Limit) (v i : H) =
        (U (v i) : H)
```

> Here `rootCornerSubspace rho n` is R, `U` is the corner isometry, and `hU` states $U \circ U = I_R$. `selfAdjoint Limit` bundles $h = h^*$; `heh` and `hhe` express the two support identities. `liftedCornerExponential` is the matrix sum defining u. In the proof, `Complex.I` is the imaginary unit; scalar pi and the representation `rho` are different objects.

Lean realization notes

Only the displayed finite family is required to have the prescribed action. The theorem does not identify the action on the whole corner with U, and does not give a small-norm bound for $u - 1$. In this Hilbert-operator setting the source adjoint $T^\dagger$ is the ordinary Hilbert adjoint T^*; the exact source notation is retained.

▼ **Exact Card identity**

Language: en
CARD_CONTENT_COMPLETE
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