The irreducible spectrum as pure-GNS representation classes

`MathlibAnnex.Analysis.CStarAlgebra.IrreducibleSpectrum`

def

Defines the irreducible spectrum using unitary equivalence classes of pure-state GNS representations.

Statement

Let A be a complex C*-algebra with compatible order, not necessarily unital or nonzero. The irreducible spectrum here is the quotient of its pure states by unitary equivalence of their GNS representations of A.

Definition

A pure state is a positive norm-one bounded complex-linear functional $\varphi : A \to \mathbb{C}$, extreme among such functionals for real convexity. Let π_φ denote the GNS representation of its canonical extension to the forced unitization, restricted to the original algebra A. The pure-state GNS irreducibility theorem guarantees irreducibility of this action.

Write $\varphi \sim \psi$ when π_φ and π_ψ are unitarily equivalent: there is a surjective complex-linear isometry W between their Hilbert spaces such that $W\pi_\varphi(a) = \pi_\psi(a)W$ for every $a \in A$. The definition is

$$\widehat{A} = \{\varphi : \varphi \text{ is a pure state of } A\}/\sim.$$

The quotient uses this unitary-equivalence relation. Equality of representation kernels is not substituted for unitary equivalence. The definition assigns no topology; the later ordinary spectrum topology is induced through the representation-kernel map, not specified here as a pure-state quotient topology.

Assumptions

Only the complex C*-algebra and its compatible order are required. The set of pure states may be empty. No auxiliary Hilbert space or separability assumption is required.

Conclusion

The result is the set of unitary equivalence classes of pure-state GNS representations. The definition permits an empty quotient when there are no pure-state representatives.

Main citations

- Exact complete definition.
- NonUnitalPureState: the exact pure-functional subtype.
- gnsRepresentation: restriction to the original algebra.
- isIrreducible_gnsRepresentation: the irreducibility guarantee.
- pureStateGNSSetoid: the entire relation and equivalence fields.

Lean source signature (exact)

The following is the complete defining right-hand side, copied from the fixed source.

```
def IrreducibleSpectrum := Quotient (pureStateGNSSetoid A)
```

In the source	Mathematical meaning
Exact declaration `NonUnitalPureState A` inside the setoid	The pure states of A defined above: positive norm-one bounded complex-linear functionals that are extreme among such functionals.
Exact declaration `pureStateGNSSetoid A`	The relation $\varphi \sim \psi$ of unitary equivalence between π_φ and π_ψ.
Exact declaration `Quotient (pureStateGNSSetoid A)`	The quotient for precisely that relation. No kernel equivalence or topology is included in this one-line RHS.

Exact surrounding source context (separate excerpts)

Source lines 23–26:

```
namespace MathlibAnnex.Analysis.CStarAlgebra

universe u
variable (A : Type u) [NonUnitalCStarAlgebra A] [PartialOrder A] [StarOrderedRing A]
```