Continuous finite conjugation relative to a class of source spaces

`ClosureSpace.HasContinuousFiniteConjugation`

def

Defines when finite conjugation is continuous in both directions.

Statement

This property requires a conjugate fgf^{-1} and its inverse to be fully continuous whenever f and g satisfy the finite-continuity and forward full-continuity conditions below. The requirement ranges over every component of every closure space in a specified class.

A closure operator here is extensive, monotone, idempotent, and sends the empty set to the empty set. It need not satisfy the finite-union axiom of topological closure. A map $h : (X, c) \to (Y, d)$ is fully continuous when

$$h(c(E)) \subseteq d(h(E)) \qquad (E \subseteq X). \tag{1}$$

Finite continuity requires (1) only for finite E. A finite-homeomorphism is a bijection for which both the map and its inverse are finitely continuous; neither is thereby assumed fully continuous.

Definition

Let (X, c) be a closure space, and let D be a class of source closure spaces in a fixed universe. For an eligible source L and $t \in L$, let $M = \mathrm{Comp}_L(t)$, equipped with its **subspace closure**. Components are generated by pairs which are not already closed, together with equality; they are not assumed to be topological connected components or closed subsets.

The property $\mathrm{FC}_D(X, c)$ holds when the following universally quantified requirement holds:

$$\begin{aligned} &\text{for every } L \in D, \quad t \in L, \quad M = \mathrm{Comp}_L(t), \\ &\text{for every finite-homeomorphism } f : M \longrightarrow X \text{ with } f \text{ fully continuous}, \\ &\text{for every finite-homeomorphism } g : M \longrightarrow M \text{ with } g \text{ fully continuous}, \\ &h = f \circ g \circ f^{-1} \text{ and } h^{-1} = f \circ g^{-1} \circ f^{-1} \text{ are fully continuous on } (X, c). \end{aligned} \tag{2}$$

Both conclusions are part of the defining conjunction. Explicitly, for every $E \subseteq X$ they require

$$\begin{aligned} h(c(E)) &\subseteq c(h(E)), \\ h^{-1}(c(E)) &\subseteq c(h^{-1}(E)). \end{aligned} \tag{3}$$

The finite-conjugation construction uses the same outer maps f and f^{-1} in the inverse, not independently chosen maps.

The hypotheses in (2) impose full continuity on f and g only. They do not impose full continuity on f^{-1} or g^{-1}. Both full-continuity conclusions in (3) are required by the definition.

Assumptions

To form this property one supplies the target closure operator c and the eligible class D. In each test, L is any member of D, t is any point, and f, g are any finite-homeomorphisms satisfying the two stated forward full-continuity conditions. No eligible source, point, or map is existentially selected by the definition. If there are no qualifying tests, the universal requirement holds vacuously; the definition asserts no realization or existence theorem.

If $j : M \hookrightarrow L$ denotes the inclusion, the source closure used in both map hypotheses is

$$c_M(E) = j^{-1}\big(c_L(j(E))\big) \qquad (E \subseteq M). \qquad (4)$$

It is not replaced by a closure generated only by finite sets, and M is not equipped with an independently chosen topology.

Conclusion

The property $\mathrm{FC}_D(X, c)$ is defined by all of (2), including both lines of (3); it need not hold for every closure space. When it holds, the full conjugation homeomorphism uses exactly fgf^{-1} and its displayed inverse.

Main citations

Full versus finite continuity supplies (1). The finite conjugate and its inverse supplies the two maps in (2). The whole defining right-hand side is at lines 41–50; packaging that same conjugate is separately at lines 54–91. Subspace closure gives (4), and the separate C^*-realizability predicate specifies the eligible class in the application.

Lean source signature (exact)

For this definition the exact excerpt includes the **entire right-hand side**. The following declaration is separate.

```
def HasContinuousFiniteConjugation (c : ClosureSpace α) (D : ClosureCat.{s} → Prop) : Prop :=
  ∀ (L : ClosureCat.{s}), D L → ∀ t : L,
    ∀ f : FiniteHomeomorph (L.str.subspace (L.str.component t)) c,
      (L.str.subspace (L.str.component t)).Continuous c f.toEquiv →
      ∀ g : FiniteHomeomorph (L.str.subspace (L.str.component t))
          (L.str.subspace (L.str.component t)),
        (L.str.subspace (L.str.component t)).Continuous
          (L.str.subspace (L.str.component t)) g.toEquiv →
        c.Continuous c (f.conjugate g).toEquiv ∧
          c.Continuous c (f.conjugate g).toEquiv.symm
```

In the source	Mathematical meaning
Target closure and source class `def HasContinuousFiniteConjugation (c : ClosureSpace α) (D : ClosureCat.{s} → Prop) : Prop :=`	The input `c` is the closure on X (the source calls its underlying type `α`). The other input specifies the class D of source closure spaces. This defines the condition $\mathrm{FC}_D(X, c)$; it does not assert that the condition holds.
All source spaces and all components `∀ (L : ClosureCat.{s}), D L → ∀ t : L,`	Every closure space $L \in D$ and every point $t \in L$ is tested. The component used below is $M = \mathrm{Comp}_L(t)$ with the closure induced from L, as in (4). No particular source space or component is selected.
The comparison and its forward continuity `∀ f : FiniteHomeomorph (L.str.subspace (L.str.component t)) c, (L.str.subspace (L.str.component t)).Continuous c f.toEquiv →`	The first map is any finite homeomorphism $f : M \to X$. Both occurrences of `L.str.subspace (L.str.component t)` are the same induced closure c_M. Separately, for every $E \subseteq M$, $$f(c_M(E)) \subseteq c(f(E)).$$ Thus the assumption is forward full continuity of this same f, not of f^{-1}.
The source automorphism and its forward continuity `∀ g : FiniteHomeomorph (L.str.subspace (L.str.component t)) (L.str.subspace (L.str.component t)), (L.str.subspace (L.str.component t)).Continuous (L.str.subspace (L.str.component t)) g.toEquiv →`	The second map is any finite homeomorphism $g : M \to M$ on that same component with that same induced closure. It must additionally satisfy, for every $E \subseteq M$, $$g(c_M(E)) \subseteq c_M(g(E)).$$ Full continuity of g^{-1} is not assumed.
Continuity of the conjugate `c.Continuous c (f.conjugate g).toEquiv ∧`	The equivalence `f.conjugate g` has underlying map $h = f \circ g \circ f^{-1} : X \to X$. The first conjunct requires, for every $E \subseteq X$, $$h(c(E)) \subseteq c(h(E)).$$
Continuity of the same conjugate inverse `c.Continuous c (f.conjugate g).toEquiv.symm`	`symm` gives the inverse of that same equivalence, namely $h^{-1} = f \circ g^{-1} \circ f^{-1}$. The second conjunct requires, for every $E \subseteq X$, $$h^{-1}(c(E)) \subseteq c(h^{-1}(E)).$$ The two conjuncts together are (3).
Finite-homeomorphism fields `structure FiniteHomeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where finitelyContinuous_toFun : c.FinitelyContinuous d toFun finitelyContinuous_invFun : d.FinitelyContinuous c invFun`	The inherited equivalence supplies a bijection and its inverse. Both fields impose closure continuity only on finite subsets, as defined in the Statement. These fields are used for $f : M \to X$ and $g : M \to M$; their additional full-continuity assumptions are separate.

In the source	Mathematical meaning

The conjugate and its actual inverse
`def FiniteHomeomorph.conjugate (f : FiniteHomeomorph c d)`
`  (g : FiniteHomeomorph c c) : FiniteHomeomorph d d :=`
`  f.symm.trans (g.trans f)`
	The conjugate is formed from the same finite homeomorphisms f and g. Composition sends $x \in X$ first by f^{-1}, then by g, then by f. Its inverse uses g^{-1} between those same outer maps, as displayed in (2).

Exact surrounding source context (separate excerpts)

These are independent ambient excerpts.

```
universe u v w s t
```

```
variable {α : Type u} {β : Type v} {γ : Type w}
variable {c : ClosureSpace α} {d : ClosureSpace β} {k : ClosureSpace γ}
```

The selected definition uses the carrier `α` and explicitly supplies its own `c`, together with source universe `s`. The surrounding `β`, `γ`, `d`, and `k` serve later transport declarations and are not additional inputs to this definition. No following declaration's local hypotheses are imported into its scope.

Related exact source for the reading guide

The conjugate and its actual inverse · Fixed source

```
def FiniteHomeomorph.conjugate (f : FiniteHomeomorph c d)
  (g : FiniteHomeomorph c c) : FiniteHomeomorph d d :=
  f.symm.trans (g.trans f)
```

Finite-homeomorphism fields · Fixed source

```
structure FiniteHomeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where
  finitelyContinuous_toFun : c.FinitelyContinuous d toFun
  finitelyContinuous_invFun : d.FinitelyContinuous c invFun
```