A finite-row average centralizes its matrix stage

`MathlibAnnex.CStarAlgebra.CAR.ofStage_commute_rowAverage`

theorem

Turns an arbitrary element of the completed CAR algebra into one commuting with a prescribed finite matrix stage.

Statement

Let A be the completed CAR algebra obtained from $A_n = M_{2^n}(\mathbb{C})$ under $a \mapsto a \otimes I_2$, and write $\iota_n : A_n \to A$ for the stage embedding. Set $d = 2^n$ and $e_{ij} = \iota_n(E_{ij})$, where E_{ij} are the standard matrix units, indexed from 0 to $d-1$. Define $\Phi_n(b) = \sum_{i=0}^{d-1} e_{i0} b e_{0i}$. Then $\iota_n(c)\Phi_n(b) = \Phi_n(b)\iota_n(c)$ for every $c \in A_n$ and every $b \in A$.

Assumptions

The integer $n \geq 0$ is arbitrary. Both c and b are arbitrary elements of the indicated algebras; neither positivity nor self-adjointness is assumed. The matrix units belong to one fixed stage and satisfy $e_{ij}e_{kl} = \delta_{jk}e_{il}$, $e_{ij}^* = e_{ji}$, and $\sum_i e_{ii} = 1$.

Conclusion

The range of Φ_n lies in the relative commutant of $\iota_n(A_n)$ inside A, namely the set of elements commuting with every member of that stage. The same finite-row formula works for every $b \in A$ once n is fixed.

Proof route

First test commutation against a matrix unit e_{kl}. Multiplying the defining sum on the left leaves only its term with index l, while multiplying on the right leaves only its term with index k. Both products are $e_{k0}be_{0l}$. Every matrix c is the finite linear combination $\sum_{k,l} c_{kl}E_{kl}$, so bilinearity gives the assertion for $\iota_n(c)$.

Proof steps

1. Use the matrix-unit relations to compute $e_{kl}\Phi_n(b) = e_{k0}be_{0l}$.
2. The same relations give $\Phi_n(b)e_{kl} = e_{k0}be_{0l}$. Thus every matrix unit commutes with $\Phi_n(b)$.
3. Expand $\iota_n(c) = \sum_{k,l} c_{kl}e_{kl}$ and distribute multiplication through the finite sums and scalar multiples.

Main citations

- Exact declaration and proof · Exact source
- MathlibAnnex.CStarAlgebra.CAR.Limit · Exact source
- MathlibAnnex.CStarAlgebra.CAR.matrixUnit · Exact source
- MathlibAnnex.CStarAlgebra.CAR.limitMatrixUnit · Exact source
- MathlibAnnex.CStarAlgebra.CAR.limitMatrixUnit_mul · Exact source
- MathlibAnnex.CStarAlgebra.CAR.star_limitMatrixUnit · Exact source
- MathlibAnnex.CStarAlgebra.CAR.sum_limitMatrixUnit_diag · Exact source
- MathlibAnnex.CStarAlgebra.CAR.rowAverageLinear · Exact source
- MathlibAnnex.CStarAlgebra.CAR.limitMatrixUnit_commute_rowAverage · Exact source

- MathlibAnnex.CStarAlgebra.CAR.ofStage_eq_sum_smul_limitMatrixUnit · Exact source
- MathlibAnnex.CStarAlgebra.CAR.rowAverageLinear_one · Exact source
- MathlibAnnex.CStarAlgebra.CAR.rowAverageLinear_nonneg · Exact source
- MathlibAnnex.CStarAlgebra.CAR.norm_rowAverageLinear_le · Exact source

Lean source signature (exact)

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theorem ofStage_commute_rowAverage (n : ℕ) (c : Stage n) (b : Limit) :
    ofStage n c * rowAverageLinear n b = rowAverageLinear n b * ofStage n c
```

> Here `Limit` is A, `Stage n` is $M_{2^n}(\mathbb{C})$, `ofStage n c` is $\iota_n(c)$, and `rowAverageLinear n b` is $\Phi_n(b)$. The source equality is precisely the commutation identity in the Statement.

Lean realization notes

The formula is a corner-based finite-row average, not an average over a unitary group. Separate cited results show $\Phi_n(1) = 1$, positivity, and the dimension-independent bound $\|\Phi_n(b)\| \leq 4\|b\|$. The present theorem establishes exact commutation with this stage; it does not assert that $\Phi_n(b)$ lies in the center of all of A.

▼ **Exact Card identity**

Language: en

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