Compact coordinates from the inverse ideal-sum map

`MathlibAnnex.Analysis.CStarAlgebra.ClosedStarIdeal.compactC0SumEquivOfDense`

def

Expresses an algebra in compact-operator coordinates through its orthogonal ideal decomposition.

Statement

Let $J = (I_i)_{i \in \iota}$ be a pairwise orthogonal family of closed $*$-ideals whose finite sums are norm dense in a complex C*-algebra A. For each ideal I_i, fix a $*$-isomorphism $E_i : I_i \to \mathcal{K}(H_i)$, where H_i is the specified complete complex Hilbert space. This definition gives a $*$-isomorphism from A to the compact-operator c_0-sum.

Definition

Let $S_J : \bigoplus_i^{c_0} I_i \to A$ be the specified ideal-sum equivalence. Let E_* apply the given E_i in every coordinate, with the same index retained. The definition is precisely

$$D = E_* \circ S_J^{-1} : A \longrightarrow \bigoplus_i^{c_0} \mathcal{K}(H_i).$$

The target is the compact-operator c_0-sum. The coordinate-evaluation theorem gives

$$(D(a))_i = E_i\big((S_J^{-1}(a))_i\big).$$

The inverse identification, single-coordinate formula, and finite-truncation formula give

$$D^{-1}(\mathrm{single}_i\, T) = \iota_i(E_i^{-1}(T)), \qquad D^{-1}(\mathrm{truncate}_F\, T) = \sum_{i \in F} \iota_i(E_i^{-1}(T_i)).$$

Here ι_i is the ideal inclusion; truncation retains precisely the finite coordinate set F. The cited evaluation theorems give these formulas.

Assumptions

Orthogonality means that products from distinct ideals vanish. Density concerns their finite sums in the ambient algebra. The coordinate maps E_i are given equivalences onto all compact operators on their respective H_i, not merely faithful compact-valued actions. No countability or nontriviality is added.

Conclusion

Thus D first resolves an element into its ideal coordinates and then applies E_i in each coordinate. Its target contains no off-diagonal operator entries and is not asserted to be the full compact-operator algebra of a Hilbert sum.

Main citations

- Exact complete defining RHS.
- c0SumEquivOfDense: complete ideal-sum equivalence.
- eval_compactC0SumEquivOfDense: coordinate formula.
- compactC0SumEquivOfDense_symm_apply: inverse identification.
- compactC0SumEquivOfDense_symm_single: one coordinate.
- compactC0SumHom_truncate: finite coordinates.

Lean source signature (exact)

Complete contiguous definition from the fixed source:

```
def compactC0SumEquivOfDense (hdense : Dense (finiteSums J)) :
    A ≃⋆ₐ[ℂ] CompactOperatorC0Sum H :=
  (c0SumEquivOfDense J hJ hdense).symm.trans
    (C0DirectSum.equiv (Fiber J)
      (fun i => ContinuousLinearMap.compactOperatorStarSubalgebra ℂ (H i))
      (Equiv.refl ι) E)
```

In the source	Mathematical meaning
Exact declaration `(c0SumEquivOfDense J hJ hdense).symm`	The inverse S_J^{-1}, directed from A to the ideal-coordinate c_0-sum.
Exact declaration `.trans (C0DirectSum.equiv ... (Equiv.refl ι) E)`	First use S_J^{-1}, then apply E_i coordinatewise, retaining the same index i.
Exact declaration `Fiber J i` and `compactOperatorStarSubalgebra ℂ (H i)`	The ideal I_i in the family $J = (I_i)_{i\in\iota}$ and the full compact-operator algebra on H_i, respectively.
Exact declaration `hdense : Dense (finiteSums J)`	The density assumption used by the ideal-sum equivalence, not a new conclusion of this definition.

Exact surrounding source context (separate excerpts)

Source lines 51–58:

```
variable {ι : Type uι} {A : Type uA} [NonUnitalCStarAlgebra A]
variable (J : ι → ClosedStarIdeal A) (hJ : PairwiseOrthogonal J)
variable (H : ι → Type uH)
variable [∀ i, NormedAddCommGroup (H i)] [∀ i, InnerProductSpace ℂ (H i)]
variable [∀ i, CompleteSpace (H i)]
variable (E : ∀ i, Fiber J i ≃⋆ₐ[ℂ]
  ContinuousLinearMap.compactOperatorStarSubalgebra ℂ (H i))
```