The geometric structure closure

`NormedSpace.geometricStructureClosure`

def

The geometric structure closure

Statement

The geometric structure closure equips the set of geometric points with the hull–kernel rule. It closes a collection of points by intersecting them as subsets of the original vector space and then taking all geometric points containing that intersection.

Definition

Let E be a normed space over a real-or-complex scalar field $\mathbb{K}$, and let E^* denote its continuous $\mathbb{K}$-linear dual. Write $B_E = \{x : \|x\| \le 1\}$. A maximal proper real convex face M of B_E is a nonempty convex subset, different from B_E, that contains both endpoints of any segment in B_E whose interior meets M, and is maximal among proper convex faces.

Put

$$\begin{aligned} \Sigma_{\mathbb{K}}(M) &= \{f \in E^* : \|f\| \le 1,\ f|_M = 1\}, \\ I_{\mathbb{K}}(M) &= \bigcup_{f \in \Sigma_{\mathbb{K}}(M)} \ker f. \end{aligned}$$

Here $f|_M = 1$ means $f(x) = 1$ for every $x \in M$.

Let $S_{\mathbb{K}}(E)$ be the set of resulting subsets $I_{\mathbb{K}}(M)$ as M ranges over these faces. Equal subsets are the same geometric point.

For $\mathscr{A} \subseteq S_{\mathbb{K}}(E)$, define

$$\begin{aligned} k_E(\mathscr{A}) &= \bigcap_{J \in \mathscr{A}} J, \\ c_E(\mathscr{A}) &= \{I \in S_{\mathbb{K}}(E) : k_E(\mathscr{A}) \subseteq I\}. \end{aligned}$$

The intersection is taken inside E, with $k_E(\varnothing) = E$, while $c_E(\mathscr{A})$ is a set of geometric points. Each geometric point is proper: any x in a representing face satisfies $f(x) = 1$ for all its supporting functionals and hence lies outside their kernel union. The cited hull–kernel construction therefore applies to this family and proves, for all $\mathscr{A}, \mathscr{B} \subseteq S_{\mathbb{K}}(E)$,

$$\mathscr{A} \subseteq c_E(\mathscr{A}), \qquad \mathscr{A} \subseteq \mathscr{B} \Longrightarrow c_E(\mathscr{A}) \subseteq c_E(\mathscr{B}),$$
$$c_E(c_E(\mathscr{A})) = c_E(\mathscr{A}), \qquad c_E(\varnothing) = \varnothing.$$

Assumptions

The space E is normed over $\mathbb{K}$, without a completeness or nontriviality hypothesis. Properness of every geometric point follows from the defining faces and is supplied internally to the hull–kernel construction.

Conclusion

The result is the closure space $(S_{\mathbb{K}}(E), c_E)$. A collection of geometric points is closed when it is fixed by the displayed operation. This operation acts on collections of points; it is distinct from norm closure inside E.

The source separately proves T_0 (distinct points have distinct singleton closures) and later T_1 (every singleton is closed), the latter using incomparability of distinct geometric points under inclusion. Finite-union preservation is not asserted here. For the zero space, the carrier is empty and the same construction applies.

Main citations

- Faces, functionals and the point family: `NormedSpace.IsMaximalFace`; the family and its properness, `NormedSpace.geometricStructureFamily` and `NormedSpace.univ_notMem_geometricStructureFamily`, lines 37–54.
- Exact closure and its membership formula: `NormedSpace.geometricStructureClosure`, lines 57–65; T_0 and the zero-space results, lines 68–86.
- The construction proving the four axioms: `ClosureSpace.HullKernel.space`, lines 69–89.
- Incomparability and singleton closedness: `NormedSpace.GeometricStructureSpace.eq_of_subset` and `NormedSpace.isT1_geometricStructureClosure`, lines 52–72.
- arXiv:2504.05551v1, Definition 4.1: the same hull–kernel formula on geometric points; the source retains the weaker normed-space assumptions stated here.

Lean source signature (exact)

The scalar and normed-space declarations used by this definition are reproduced separately below.

```
def geometricStructureClosure : ClosureSpace (GeometricStructureSpace 𝕜 E) :=
  ClosureSpace.HullKernel.space (geometricStructureFamily 𝕜 E)
    (univ_notMem_geometricStructureFamily 𝕜 E)
```

In the source	Mathematical meaning
Exact declaration `𝕜, [RCLike 𝕜]`	The real or complex scalar field $\mathbb{K}$.
Exact declaration `E, [NormedAddCommGroup E], [NormedSpace 𝕜 E]`	A normed vector space E over $\mathbb{K}$; it may be zero and need not be complete.
Exact declaration `GeometricStructureSpace 𝕜 E, geometricStructureFamily 𝕜 E`	The point set $S_{\mathbb{K}}(E)$ and its underlying family of subsets $\mathcal{F}_{\mathbb{K}}(E)$, defined by maximal-face support-kernel unions.
Exact declaration `ClosureSpace.HullKernel.space (geometricStructureFamily 𝕜 E)`	The defining construction gives $c_E(\mathscr{A}) = \{I \in S_{\mathbb{K}}(E) : \bigcap_{J \in \mathscr{A}} J \subseteq I\}$ for every subfamily $\mathscr{A}$; an empty intersection is E.
Exact declaration `univ_notMem_geometricStructureFamily 𝕜 E`	The cited properness theorem supplies $E \notin \mathcal{F}_{\mathbb{K}}(E)$. It is a proved input to the constructor, not an additional hypothesis on E.
Exact declaration `: ClosureSpace (GeometricStructureSpace 𝕜 E)`	The returned object is the operation c_E together with extensivity, monotonicity, idempotence and $c_E(\varnothing) = \varnothing$. Finite-union preservation is not asserted.

Source context (separate exact excerpt)

The surrounding declarations below are from lines 32–33; their mathematical meaning is included in the guide above.

```
variable (𝕜 : Type v) [RCLike 𝕜]
variable (E : Type u) [NormedAddCommGroup E] [NormedSpace 𝕜 E]
```

Exact declaration for the source guide

```
def geometricStructureClosure : ClosureSpace (GeometricStructureSpace 𝕜 E) :=
  ClosureSpace.HullKernel.space (geometricStructureFamily 𝕜 E)
    (univ_notMem_geometricStructureFamily 𝕜 E)

/-- The closure formula exposes the intersection, including the empty intersection. -/
```