Purity of the completed root state

`MathlibAnnex.CStarAlgebra.CAR.isPureState_rootState`

theorem

Passes extremality of the root vector states from every finite matrix stage to the completed CAR algebra.

Statement

Let A be the completed CAR algebra with stage embeddings $\iota_n : M_{2^n}(\mathbb{C}) \to A$. Let $\omega : A \to \mathbb{C}$ be the continuous linear functional characterized by $\omega(\iota_n(a)) = a_{00}$ for every stage matrix a. Then ω is a pure state: it is an extreme point of the convex set of continuous positive unital complex-linear functionals on A.

Assumptions

The embeddings come from $a \mapsto a \otimes I_2$. The index 0 is the distinguished first coordinate in each stage. The coordinate functionals are compatible and bounded by the matrix norm, so they extend to the specified functional ω on the completion. Positivity and $\omega(1) = 1$ are established before the purity argument. Convex combinations use real coefficients.

Conclusion

If φ and ψ are states on A, $0 < t < 1$, and $\omega = t\varphi + (1-t)\psi$, then $\varphi = \psi = \omega$. Thus completion has not introduced a nontrivial convex decomposition of this state.

Proof route

Restrict a proposed convex decomposition of ω to each finite matrix stage. The restriction of a state is again a state, because ι_n is unital and preserves positivity. The root coordinate state $a \mapsto a_{00}$ is pure on each matrix algebra, so both restricted states equal it. The two continuous functionals therefore agree with ω on the dense union of stages, and hence on all of A.

Proof steps

1. Write $\omega = t\varphi + (1-t)\psi$ with states φ, ψ and $0 < t < 1$.
2. For each n, compose with ι_n to obtain a convex decomposition of $a \mapsto a_{00}$ into states on $M_{2^n}(\mathbb{C})$.
3. Finite-stage purity forces $\varphi \circ \iota_n = \psi \circ \iota_n = (a \mapsto a_{00})$.
4. Continuity and density imply $\varphi = \psi = \omega$. The source uses the equivalent one-sided extreme-point criterion, establishing the first equality, which suffices.

Main citations

- Exact declaration and proof · Exact source
- MathlibAnnex.CStarAlgebra.CAR.rootState · Exact source
- MathlibAnnex.CStarAlgebra.CAR.rootState_stage · Exact source
- MathlibAnnex.CStarAlgebra.CAR.rootState_mem_stateSpace · Exact source
- MathlibAnnex.CStarAlgebra.CAR.restrictState_mem_stateSpace · Exact source

- MathlibAnnex.CStarAlgebra.CAR.isPureState_rootFunctional · Exact source
- MathlibAnnex.CStarAlgebra.CAR.eq_rootState_of_restrict · Exact source
- MathlibAnnex.CStarAlgebra.CAR.dense_stageRange · Exact source

Lean source signature (exact)

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theorem isPureState_rootState : MathlibAnnex.CStarAlgebra.IsPureState Limit rootState
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> Here `Limit` denotes the completed CAR algebra A, `rootState` denotes ω, and `IsPureState` expresses the extreme-point property stated above.

Lean realization notes

The purity asserted here is extremality in the state space. The proof uses the pure finite-stage restrictions and density; it makes no assertion that all states on A are pure.

▼ **Exact Card identity**

Language: en

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