The specified equivalence from a dense orthogonal ideal sum

`MathlibAnnex.Analysis.CStarAlgebra.ClosedStarIdeal.c0SumEquivOfDense`

def

Identifies an algebra with the c₀-sum of a dense orthogonal family of closed ideals.

Statement

Let $J = (I_i)_{i \in \iota}$ be a family of closed $*$-ideals in a complex C*-algebra A. Assume $xy = 0$ whenever $x \in I_i$, $y \in I_j$ and $i \neq j$, and that the set of finite sums of elements of these ideals is norm dense in A. The ideal-sum map defines a $*$-isomorphism from the dependent c_0-sum of these ideals onto A.

Definition

Give each ideal I_i its inherited algebra structure, with inclusion $\iota_i : I_i \hookrightarrow A$. Use the dependent c_0-sum of these ideals. The forward map is the continuous ideal-sum map

$$S_J : \bigoplus_i^{c_0} I_i \longrightarrow A.$$

The single-coordinate and finite-truncation formulas are

$$S_J(\mathrm{single}_i\, x) = \iota_i(x), \qquad S_J(\mathrm{truncate}_F\, x) = \sum_{i \in F} \iota_i(x_i).$$

Here truncation sets coordinates outside the finite set F to zero. On a general family, S_J is the continuous extension of finite ideal summation. Its injectivity and surjectivity under density make this same map a bijection. It preserves addition, multiplication, complex scalars and involution.

Assumptions

The family consists of closed $*$-ideals satisfying the stated ordered zero-product condition. Density concerns all finite sums in A, not density of each individual ideal. No countability or nonempty-index assumption is imposed.

Conclusion

The output S_J is directed from the ideal c_0-sum to A. Its inverse gives the ideal coordinates used in the subsequent compact-operator decomposition.

Main citations

- Exact complete defining RHS.

- Fiber: the ideal.
- PairwiseOrthogonal: full ordered zero-product condition.
- finiteSums: full set definition.
- c0SumHom: the specified continuous ideal-sum construction.
- c0SumHom_single: single-coordinate guarantee.
- c0SumHom_truncate: finite-truncation guarantee.
- c0SumEquivOfDense_apply: the same forward map.

Lean source signature (exact)

Complete contiguous definition from the fixed source:

```
def c0SumEquivOfDense (hJ : PairwiseOrthogonal J)
    (hdense : Dense (finiteSums J)) :
    C0DirectSum (Fiber J) ≃⋆ₐ[ℂ] A :=
  { Equiv.ofBijective (c0SumHom J hJ)
      ⟨c0SumHom_injective J hJ, c0SumHom_surjective J hJ hdense⟩ with
    map_add' := map_add (c0SumHom J hJ)
    map_mul' := map_mul (c0SumHom J hJ)
    map_smul' := map_smul (c0SumHom J hJ)
    map_star' := map_star (c0SumHom J hJ) }
```

In the source	Mathematical meaning
Exact declaration `J : ι → ClosedStarIdeal A` and `Fiber J`	The family $J = (I_i)$; `Fiber J i` is the same ideal I_i, regarded as an algebra with its inherited operations.
Exact declaration `hJ` and `hdense`	The displayed zero-product and finite-sum density assumptions.
Exact declaration `Equiv.ofBijective (c0SumHom J hJ) ...`	A bijection with exactly the existing ideal-sum map as its forward function; named results supply the two properties.
Exact declaration `map_add'`, `map_mul'`, `map_smul'`, `map_star'`	These fields equip that same equivalence with its complex $*$-algebra structure.
Exact declaration `C0DirectSum (Fiber J) ≃⋆ₐ[ℂ] A`	The direction is from the coordinate family to the ambient algebra.

Exact surrounding source context (separate excerpts)

Source lines 44–48:

```
universe uι uA

variable {ι : Type uι} {A : Type uA} [NonUnitalCStarAlgebra A]

namespace ClosedStarIdeal
```

Source lines 63–63:

```
variable (J : ι → ClosedStarIdeal A)
```