GSS rigidity when either algebra is finite-dimensional

`MathlibAnnex.Analysis.CStarAlgebra.nonempty_starAlgEquiv_iff_nonempty_geometricStructure_homeomorph_of_finiteDimensional_or`

theorem

Obtains algebraic rigidity when either algebra is finite-dimensional.

Statement

Let A, B be complex C*-algebras. For a complex C*-algebra D, write $S(D)$ for its geometric structure space. Its points are the unions of kernels of norm-one bounded complex-linear functionals supporting a maximal proper face of the closed unit ball of D; faces use real convexity. Write c_D for its hull–kernel closure:

$$c_D(E) = \left\{ I \in S(D) : \bigcap_{J \in E} J \subseteq I \right\}, \qquad E \subseteq S(D).$$

These are the geometric structure definition and its closure, with the norm-one description justified by the norm-one support characterization.

Assume that at least one of A, B is finite-dimensional as a complex vector space. Then A and B are $*$-isomorphic if and only if $S(A)$ and $S(B)$ are homeomorphic as closure spaces. Here a closure-space homeomorphism is a bijection $e : S(A) \to S(B)$ satisfying

$$e(c_A(E)) = c_B(e(E)), \qquad E \subseteq S(A).$$

The equality is required for every subset, not just finite subsets.

Assumptions

Finite-dimensionality is required on at least one side, not on both as an input. The algebras may be zero. No nonzero, unitality, separability, or prescribed dimension hypothesis is imposed.

Conclusion

The equivalence concerns existence of a $*$-algebra isomorphism and existence of a closure-space homeomorphism. It does not require the isomorphism to induce a prescribed homeomorphism, or assert uniqueness.

Main citations

The either-side finite-dimensional theorem uses the finite separating GNS representation theorem and rigidity for a given compact action. The finite separating family and compact action are supplied by these results, not additional hypotheses.

Lean source signature (exact)

Exact source and declaration. Only the literal signature is displayed here; the linked source retains the proof.

```
theorem
nonempty_starAlgEquiv_iff_nonempty_geometricStructure_homeomorph_of_finiteDimensional_or
    (hfinite : FiniteDimensional ℂ A ∨ FiniteDimensional ℂ B) :
    Nonempty (A ≃⋆ₐ[ℂ] B) ↔
      Nonempty (ClosureSpace.Homeomorph (NormedSpace.geometricStructureClosure ℂ A)
        (NormedSpace.geometricStructureClosure ℂ B))
```

In the source	Mathematical meaning

Two C*-algebras
`variable {A : Type u} {B : Type v}`
`variable [NonUnitalCStarAlgebra A]`
`[NonUnitalCStarAlgebra B]`
	The inputs A, B are arbitrary complex C*-algebras, including zero. Their size levels are independent. The source type allows unital algebras but does not require a unit.

Finite-dimensionality on either side
`(hfinite : FiniteDimensional ℂ A ∨`
`FiniteDimensional ℂ B) :`
	The disjunction requires $\dim_{\mathbb{C}} A < \infty$ or $\dim_{\mathbb{C}} B < \infty$. One side suffices, and no particular dimension or positivity condition is imposed.

Existence of a star-isomorphism `Nonempty (A ≃⋆ₐ[ℂ] B) ↔`	`Nonempty` says that the type of $*$-algebra isomorphisms $A \to B$ has an element. The outer equivalence relates this existence to the existence in the next row, not to a choice of a prescribed map.

Existence of a closure-space homeomorphism
`Nonempty (ClosureSpace.Homeomorph`
`(NormedSpace.geometricStructureClosure ℂ A)`
`  (NormedSpace.geometricStructureClosure ℂ B)) :=`
`by`
	The complete arguments are c_A on $S(A)$ and c_B on $S(B)$. `Nonempty` requires one homeomorphism $e : S(A) \to S(B)$ between these closure spaces.

The same map and inverse
`structure Homeomorph (c : ClosureSpace α) (d :`
`ClosureSpace β) extends α ≃ β where`
`  continuous_toFun : c.Continuous d toFun`
`  continuous_invFun : d.Continuous c invFun`
	The inherited equivalence supplies the bijection e and its inverse e^{-1}. The first field requires, for every $E \subseteq S(A)$,
	$$e(c_A(E)) \subseteq c_B(e(E)).$$
	The second requires, for every $F \subseteq S(B)$,
	$$e^{-1}(c_B(F)) \subseteq c_A(e^{-1}(F)).$$
	Both concern the same e and together express the closure equality in the Statement.

Continuity for arbitrary subsets
`def Continuous (c : ClosureSpace α) (d :`
`ClosureSpace β) (f : α → β) : Prop :=`
`  ∀ s : Set α, f '' c.closure s ⊆ d.closure (f ''`
`s)`
	For a map $r : (X, c) \to (Y, d)$, continuity means
	$$r(c(E)) \subseteq d(r(E)) \quad (E \subseteq X).$$
	In the preceding row this definition is applied first to $r = e$ and then to $r = e^{-1}$, with the domain and target closures interchanged.

Exact surrounding source context (separate excerpts)

Exact surrounding source:

```
universe u v w z

variable {A : Type u} {B : Type v}
variable [NonUnitalCStarAlgebra A] [NonUnitalCStarAlgebra B]
```

Related exact source for the reading guide

Existence of a closure-space homeomorphism · Fixed source

```
      Nonempty (ClosureSpace.Homeomorph (NormedSpace.geometricStructureClosure ℂ A)
        (NormedSpace.geometricStructureClosure ℂ B)) := by
```

Continuity for arbitrary subsets · Fixed source

```
def Continuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop :=
  ∀ s : Set α, f '' c.closure s ⊆ d.closure (f '' s)
```

The same map and inverse · Fixed source

```
structure Homeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where
  continuous_toFun : c.Continuous d toFun
  continuous_invFun : d.Continuous c invFun
```