The primitive spectrum as representation kernels

`MathlibAnnex.Analysis.CStarAlgebra.PrimitiveSpectrum`

def

Defines the primitive spectrum as the set of pure-state GNS representation kernels.

Statement

Let A be a complex C*-algebra with compatible order, not necessarily unital or nonzero. The primitive spectrum here is the set of subsets of A occurring as kernels of pure-state GNS representations.

Definition

A pure state is a bounded complex-linear functional $\varphi : A \to \mathbb{C}$ that is positive, has norm one, and is extreme among positive norm-one functionals, for real convexity. Its canonical state extension to the forced unitization has a GNS representation; π_φ denotes the restriction of that representation to A.

$$\mathrm{Prim}(A) = \{\ker \pi_\varphi : \varphi \text{ is a pure state of } A\}.$$

A point is the representation kernel itself, not a preferred pure state. The ideal identification regards it as a two-sided ideal of A. This definition assigns no topology; the spectrum topology is defined separately.

Assumptions

The source requires a nonunital complex C*-algebra, a partial order and compatible order. No nontriviality, separability, faithfulness, finite dimension or chosen representation is an additional input.

Conclusion

Points are representation kernels, regarded as subsets of A. Pure states with the same GNS kernel determine the same point. This is not the quotient by unitary equivalence, nor is it the family of functional kernels.

Main citations

- Exact complete definition.
- primitiveKernelFamily: the full range expression.
- NonUnitalPureState: the pure-state subtype.
- gnsRepresentation: the action of the original algebra.
- coe_toTwoSidedIdeal: the ideal-carrier guarantee.

Lean source signature (exact)

The following is the complete defining right-hand side, copied from the fixed source.

```
def PrimitiveSpectrum := {P : Set A // P ∈ primitiveKernelFamily A}
```

In the source	Mathematical meaning
Exact declaration `A` and the surrounding three instances	The complex C*-algebra with the order used for positivity.
Exact declaration `{P : Set A // ...}`	A subset of the original algebra together with membership in the indicated family.
Exact declaration `P ∈ primitiveKernelFamily A`	$P = \ker \pi_\varphi$ for some pure state φ of A.
The complete one-line RHS **Exact declaration**	No quotient of representations and no topological data are fields of this definition.

Exact surrounding source context (separate excerpts)

Source lines 22–26:

```
namespace MathlibAnnex.Analysis.CStarAlgebra

universe u v
variable (A : Type u) [NonUnitalCStarAlgebra A] [PartialOrder A] [StarOrderedRing A]
open NonUnitalCStarRepresentation
```