An inequivalent GNS fiber has no residual common range

`MathlibAnnex.CStarAlgebra.CAR.selected_commonFixedProjection_eq_zero`

theorem

Uses compression and cyclic transport to exclude fixed vectors in every other chosen pure-state class.

Statement

Fix a shell family and indices $i, j \in I$ as below, with $j \neq i$. For $f_{i,n} = \alpha_i(f_n)$, let $F_j = \bigcap_{n\geq 0} \operatorname{ran} \pi_j(f_{i,n})$ and let P_j be its orthogonal projection. Then $P_j = 0$, or equivalently $F_j = \{0\}$.

Assumptions

Let A be the completed CAR algebra, the completion of the matrix stages $M_{2^n}(\mathbb{C})$ under $a \mapsto a \otimes I_2$. Write f_n for the image of the first diagonal matrix unit. These are decreasing projections with $f_0 = 1$. Write ι_n for the canonical embedding of stage n. The root state ω is determined by $\omega(\iota_n(a)) = a_{00}$ on every stage.

Choose one pure state φ_j from each unitary-equivalence class $j \in I$ of pure-state Gelfand-Naimark-Segal (GNS) representations, retaining ω at the root index o. Write (H_j, π_j, ξ_j) for its complete complex GNS space, unital star representation and unit cyclic vector. Thus $\varphi_j(a) = \langle \xi_j, \pi_j(a)\xi_j \rangle$, and the vectors $\pi_j(a)\xi_j$ are dense in H_j. Inner products are linear in the second argument.

A fixed shell family supplies unital complex star automorphisms α_j and elements $w_{j,n} \in A$ with $\varphi_j \circ \alpha_j = \omega$, $w_{j,n}^* w_{j,n} = \alpha_j(f_n - f_{n+1})$, and $w_{j,n} w_{j,n}^* = f_n - f_{n+1}$. At the root, α_o is the identity and $w_{o,n} = f_n - f_{n+1}$. Fix $i \in I$ and put $f_{i,n} = \alpha_i(f_n)$. The family is supplied; its existence is not an extra conclusion here. The condition $j \neq i$ means that the unitary-equivalence classes are distinct, not merely that two state functionals in the same class differ. The selected representation π_j is irreducible and has nonzero Hilbert space because $\|\xi_j\| = 1$.

Conclusion

The transported flag leaves no nonzero vector fixed in an inequivalent chosen fiber. As a separate consequence of decreasing orthogonal projections, $\pi_j(f_{i,n})x \to 0$ in norm for each fixed $x \in H_j$. Thus the surviving line in the matching fiber is sharply separated from every other chosen class.

Proof route

If F_j contained a unit vector η, the transported CAR compression estimate $\|f_{i,n} b f_{i,n} - \varphi_i(b) f_{i,n}\| \to 0$, valid for every $b \in A$, would imply $\langle \eta, \pi_j(b)\eta \rangle = \varphi_i(b)$. Irreducibility makes η cyclic. Equality of this vector state with the state of ξ_i yields a unitary intertwiner from π_i to π_j, contradicting their distinct classes.

Proof steps

1. If $P_j \neq 0$, choose z with $P_j z \neq 0$ and normalize $P_j z$ to a unit vector η. Since P_j projects onto the common range, every $\pi_j(f_{i,n})$ fixes η.

2. Take the matrix coefficient of the represented compression error against η. Fixedness and unit norm turn it into the constant $\langle \eta, \pi_j(b)\eta \rangle - \varphi_i(b)$; norm convergence forces this constant to vanish.

3. The orbit closures of ξ_i and η are their full Hilbert spaces. Equality of vector states gives equality of orbit inner products, so the map $\pi_i(b)\xi_i \mapsto \pi_j(b)\eta$ extends to a surjective complex-linear isometry that intertwines the representations. This contradicts $j \neq i$.

Main citations

- Exact declaration and proof · Exact source
- MathlibAnnex.CStarAlgebra.CAR.RepresentativeShellFamily · Exact source
- MathlibAnnex.CStarAlgebra.CAR.transportedFlag · Exact source
- MathlibAnnex.CStarAlgebra.CAR.tendsto_representative_transported_compression · Exact source
- MathlibAnnex.CStarAlgebra.PureState.isIrreducible_selectedRepresentation · Exact source
- MathlibAnnex.CStarAlgebra.PureState.no_unitaryIntertwiner_selectedRepresentation · Exact source
- MathlibAnnex.Analysis.CStarAlgebra.commonFixedProjection_eq_zero_of_no_unitary · Exact source
- StarAlgHom.existsUnique_pointedCyclicTransport · Exact source
- Submodule.tendsto_starProjection_iInf · Exact source
- The root state bundled with its purity proof · Exact source

Lean source signature (exact)

```
theorem selected_commonFixedProjection_eq_zero
    (family : RepresentativeShellFamily) {i j : MathlibAnnex.CStarAlgebra.PureState.GNSClass Limit}
    (hij : j ≠ i) :
    commonFixedProjection (fun n ↦
      MathlibAnnex.CStarAlgebra.PureState.selectedRepresentation completedRootPureState j
        (transportedFlag family i n)) = 0
```

> Here `Limit` is A and `completedRootPureState` is the root state ω, bundled with its purity proof. `transportedFlag family i n` is $f_{i,n}$, and `selectedRepresentation completedRootPureState j` is π_j. Thus `i` selects the flag, while `j` selects the representation on which it acts. The hypothesis `hij : j ≠ i` excludes the matching GNS class. `commonFixedProjection` is P_j, so `= 0` is the operator equality in the Statement. The quantified indices range over chosen GNS classes, not over arbitrary vectors or arbitrary representations.

Lean realization notes

Irreducibility is used to make a nonzero candidate fixed vector cyclic. The conclusion is not asserted for an arbitrary reducible representation or for a different vector state in the same GNS class. The exact source proves equality of projections; strong convergence uses the additional cited projection theorem and does not imply operator-norm convergence.

▼ **Exact Card identity**

Language: en
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Exposition revision: 1 · SHA-256: 9846526d3fb298d1a42dfe59f1e3ec37ec375206e8bc24f5f86e04f367e223f1
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