Two-sided inner sequences imply pure-state homogeneity

`MathlibAnnex.CStarAlgebra.CAR.homogeneity_of_innerIntertwiningSequences`

theorem

The forward and inverse limits close the homogeneity argument once an intertwining sequence is supplied for every pure-state pair.

Statement

Let A be the completed CAR algebra, the norm completion of the matrix stages $A_n = M_{2^n}(\mathbb{C})$ under $a \mapsto a \otimes I_2$, with the fixed coordinate reindexing. A state means a positive continuous complex-linear functional taking value 1 at the identity; a pure state is an extreme point of this convex state space. Assume that every pair of pure states φ, ψ admits inner automorphisms α_n such that $\alpha_n(a)$ and $\alpha_n^{-1}(a)$ are norm-Cauchy for every a, and $\varphi(\alpha_n(a)) \to \psi(a)$. Then there is an automorphism α with $\varphi \circ \alpha = \psi$, and for every finite $F \subset A$ and $\varepsilon > 0$ a unitary u satisfies $\|\alpha(a) - uau^*\| < \varepsilon$ for every $a \in F$.

Assumptions

The supplier assumption quantifies over all pure-state pairs and gives the sequence conditions just stated. The theorem is conditional on this supplier. Completeness is a property of the completed CAR algebra, not a claim that a merely dense algebraic union already has these limits.

Conclusion

The algebra has the pure-state homogeneity property. The inverse Cauchy condition is what supplies surjectivity of the limit; approximate innerness then follows by choosing one sufficiently late term for each finite set.

Proof route

Fix a supplied sequence and define $\alpha(a) = \lim_n \alpha_n(a)$ and $\beta(a) = \lim_n \alpha_n^{-1}(a)$. Operations pass through the limits, and isometry lets the exact inverse identities pass through moving inputs. Thus α and β are mutually inverse star homomorphisms. Continuity transports the state equation to the limit, and a common late index yields the finite-set approximation.

Proof steps

1. Each automorphism is an isometry. Norm completeness and the two Cauchy hypotheses therefore give the two pointwise limits. Continuity of addition, multiplication, scalar multiplication, and involution makes the forward limit a unital complex star homomorphism; the inverse limit has the same properties.

2. For fixed a, use $\alpha_n(\alpha_n^{-1}(a)) = a$. The difference between $\alpha_n(\beta(a))$ and this constant value has norm $\|\beta(a) - \alpha_n^{-1}(a)\| \to 0$. Passing to the limit at the fixed input $\beta(a)$ gives $\alpha(\beta(a)) = a$. Interchanging the two sequences gives $\beta(\alpha(a)) = a$. This proves bijectivity rather than only injectivity.

3. Continuity of φ gives $\varphi(\alpha_n(a)) \to \varphi(\alpha(a))$. The supplied state limit and uniqueness of limits imply $\varphi(\alpha(a)) = \psi(a)$.

4. For a finite set F, choose a single index N beyond the convergence thresholds for all $a \in F$. If $\alpha_N = \mathrm{Ad}(u_N)$, then $\|\alpha(a) - u_N a u_N^*\| < \varepsilon$ on F. For the empty set the requirement is vacuous.

Main citations

- The stated existence or structural result · Exact source
- The precise supplier condition · Exact source
- Construction of mutually inverse limits · Exact source
- Diagonal isometry argument giving bijectivity · Exact source
- State equation and simultaneous finite-set approximation · Exact source

Lean source signature (exact)

```
theorem homogeneity_of_innerIntertwiningSequences
    (hlocal : ∀ (phi psi : Limit →L[ℂ] ℂ),
      MathlibAnnex.CStarAlgebra.IsPureState Limit phi → MathlibAnnex.CStarAlgebra.IsPureState Limit psi →
        HasInnerIntertwiningSequence phi psi) :
    PureStateHomogeneity
```

> Here `hlocal` supplies the sequence for each pair of pure states. In the linked proof, rather than in this signature, `f` is (α_n), and `hf`, `hfinv` are its forward and actual inverse Cauchy hypotheses. The exact proof uses `twoSidedPointwiseLimit` and its state/approximation theorem. Its output `PureStateHomogeneity` has the composition direction $\varphi \circ \alpha = \psi$.

Lean realization notes

This assembly theorem does not itself construct the sequences. The separate pure-state supplier discharges that hypothesis for CAR. It neither uses nor proves the general all-simple-separable-algebras KOS proposition.

▼ **Exact Card identity**

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