Finite-dimensional space for a unital representative of the unique irreducible-representation class

`MathlibAnnex.Analysis.CStarAlgebra.Representation.finiteDimensional_space_of_singleton`

Shows that the Hilbert space of a separably acting unital representative of the unique irreducible-representation class is finite-dimensional.

Statement

Let A be a nonzero unital complex C*-algebra, H a separable complex Hilbert space, and π a representation of A on H that is a representative of the unique unitary-equivalence class of nonzero irreducible representations of A. H is finite-dimensional over ℂ. The conclusion is conditional on the unique-class hypothesis, not a statement about all unital irreducible representations.

Assumptions

Let A be a nonzero unital complex C*-algebra, H a separable complex Hilbert space, and π a representation of A on H that is a representative of the unique unitary-equivalence class of nonzero irreducible representations of A.

Conclusion

H is finite-dimensional over ℂ. The conclusion is conditional on the unique-class hypothesis, not a statement about all unital irreducible representations.

Proof idea

The source first obtains injectivity and simplicity from the unique-class hypothesis for π, then a nonzero scalar-corner projection with compact image. The compact-preimage ideal is nonzero and closed, hence all of A by simplicity. Thus π(1)=id_H is compact, and the compact-identity criterion makes H finite-dimensional.

Proof steps

1. Exact Lean statement: ∀ {A : Type u} [inst : CStarAlgebra A] [inst_1 : PartialOrder A] [StarOrderedRing A] {H : Type v} [inst_3 : NormedAddCommGroup H] [inst_4 : InnerProductSpace ℂ H] [inst_5 : CompleteSpace H] [Nontrivial A] [TopologicalSpace.SeparableSpace H] (pi : MathlibAnnex.Analysis.CStarAlgebra.Representation A H), pi.IsSingletonIrreducibleModel → FiniteDimensional ℂ H

2. FiniteDimensional ℂ H.

3. The source first obtains injectivity and simplicity from the unique-class hypothesis for π, then a nonzero scalar-corner projection with compact image. The compact-preimage ideal is nonzero and closed, hence all of A by simplicity. Thus π(1)=id_H is compact, and the compact-identity criterion makes H finite-dimensional.

Main citations

MathlibAnnex.Analysis.CStarAlgebra.IsMaximalAbelian

Exact formal dependency; inspect the linked Card and exact source.

MathlibAnnex.Analysis.CStarAlgebra.Representation.isSimpleCStarAlgebra_of_singleton_of_injective

Exact formal dependency; inspect the linked Card and exact source.

MathlibAnnex.Analysis.CStarAlgebra.exists_pureState_nonzero_on_star_mul_self

Exact formal dependency; inspect the linked Card and exact source.

MathlibAnnex.CStarAlgebra.compactPreimageIdeal

Exact formal dependency; inspect the linked Card and exact source.

MathlibAnnex.Topology.exists_isOpen_singleton

Exact formal dependency; inspect the linked Card and exact source.

Lean source signature (exact)

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theorem finiteDimensional_space_of_singleton [Nontrivial A]
[TopologicalSpace.SeparableSpace H]
(pi : Representation A H)
(hsingle : IsSingletonIrreducibleModel.{u, v, u} pi) :
FiniteDimensional ℂ H
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Read exact MathlibAnnex v0.4.0 source

Exact Card identity

Stable Card ID: 956f7a6ee326f4d3fe0fff3198e6dde0fb0903ec6f51cfb6ad39f61c2663e298

Card SHA-256: 88f0d99a8e018be7d631ec67a0b283306e74549bc5526e3185861f3927db6411

Approved exposition SHA-256: 152b7b786a72e69d25101af19a2e68178113389e2a9031ccdebc1acafc203f13

AI-assisted material. Selected, edited, and reviewed by Ryotaro Tanaka. The recorded review is source–exposition correspondence, not an independent proof audit.