

A Naimark counterexample of norm density continuum

`MathlibAnnex.CStarAlgebra.CAR.existsNaimarkCounterexampleOfDensity_continuum`

Supplies an unconditional witness to the general density-existence statement.

Statement

There exists a nontrivial complex C^* -algebra D of exact norm density $\mathfrak{c} = 2^{\aleph_0}$ with a nonzero irreducible representation π such that every nonzero irreducible representation of D is unitarily equivalent to π and π is not a model of the entire compact-operator algebra.

Assumptions

No CH assumption is made. The existence predicate permits nonunital C^* -algebras and nonunital complex $*$ -representations. In this closed statement, the algebra carrier, displayed Hilbert space, and comparison Hilbert spaces lie in Type (universe zero).

Conclusion

The density-existence predicate holds at $\mathfrak{c}$. A witness is the fixed unital CAR-based algebra A with its atomic inclusion; its density is exactly $\mathfrak{c}$.

Proof route

Insert the fixed A and atomic representation into the general possibly nonunital existence predicate.

Proof steps

1. Use the atomic inclusion as a nonunital-API representation without changing its action.
2. Convert the proved singleton property to the general nonunital representation predicate.
3. Use the compact-model exclusion and $\text{dens}(A) = \mathfrak{c}$.

Lean realization notes

The displayed predicate explicitly negates `IsCompactOperatorModel` of the displayed π . In ordinary C*-algebraic language this yields a counterexample; it is not a silently added quantified condition in the Lean definition. The predicate has no faithful-separable-representation requirement. Its witness here is nonetheless the same fixed A . This closed existence theorem is not being promoted to an all-universe statement.

Language: en · CARD_CONTENT_COMPLETE

Main citations

[MathlibAnnex.CStarAlgebra.CAR.hasDensityCharacter_atomicCounterexampleAlgebra](#)

[MathlibAnnex.Analysis.CStarAlgebra.Representation.IsSingletonIrreducibleModelAmongNonUnital.isSingletonIrreducibleModel_toNonUnitalStarAlgHom](#)

Lean source declaration (exact)

```
-- The fixed construction supplies an ordinary counterexample of exact
norm density continuum. This existence assertion has no CH premise. -/
theorem existsNaimarkCounterexampleOfDensity_continuum :
  ExistsNaimarkCounterexampleOfDensity (Cardinal.continuum : Cardinal.{0}) := by
  refine {AtomicCounterexampleAlgebra, inferInstance, inferInstance, inferInstance,
    inferInstance,
    MathlibAnnex.CStarAlgebra.PureState.SelectedAtomicHilbert completedRootPureState,
    inferInstance, inferInstance, inferInstance,
    (shellFamilyInclusion homogeneityShellFamily).toNonUnitalStarAlgHom, ?_, ?_,
    hasDensityCharacter_atomicCounterexampleAlgebra}
  · exact
    Representation.IsSingletonIrreducibleModelAmongNonUnital.isSingletonIrreducibleModel_toNonUnitalStarAlgHom
    (isUniqueIrreducibleModelAmongNonUnital_shellFamilyInclusion.{0}
      homogeneityShellFamily).2
  · exact not_isCompactOperatorModel_shellFamilyTarget homogeneityShellFamily _
```

[Read exact MathlibAnnex v0.4.0 source](#)

Exact Card identity

Stable Card ID: 9e4fa34878df2ba18cbc8247c4d9da4877ccd5abd32747c76de2e936615fa461

Card SHA-256: 228ba19d3e6f019d7447205532137a8eabdb20491d59bbae4fa507ce55cb654f

Approved exposition SHA-256: b89acbcc403dac8b1137673529218f8b42524b5ad7731e35de2ef339db1fa604

AI-assisted material. Selected, edited, and reviewed by Ryotaro Tanaka. The recorded review is source-exposition correspondence, not an independent proof audit.