A Naimark counterexample of continuum norm density

`MathlibAnnex.CStarAlgebra.CAR.existsNaimarkCounterexampleOfDensity_continuum`

theorem

Gives a C*-algebra of norm density $\mathfrak{c}$ whose nonzero irreducible representations form one unitary-equivalence class, without an identification with the compact operators.

Statement

There exist a nonzero complex C*-algebra B, not required to be unital, a complex Hilbert space K, and a nonzero irreducible $*$-representation $\sigma : B \to B(K)$ with the following properties. Every nonzero irreducible $*$-representation of B is unitarily equivalent to σ. The representation σ is not both injective and surjective onto $\mathcal{K}(K)$, the full algebra of compact operators on K. Finally,

$$\mathrm{dens}_{\|\cdot\|}(B) = \mathfrak{c}, \qquad \mathfrak{c} = 2^{\aleph_0}.$$

Here density refers to the norm topology. None of the representations is required in advance to preserve an identity.

Assumptions

This is an existence theorem with no CH hypothesis. Its statement allows nonunital C*-algebras and places no requirement on the existence of a faithful representation on a separable Hilbert space.

Conclusion

One may take $B = T = C^*(\pi(A) \cup \{L_i : i \in I\})$, where A is CAR, π is the selected GNS direct sum, and L_i are the unitaries from the fixed homogeneity and shell construction. Take $K = \mathcal{H}$ and $\sigma = \rho_0$, the inclusion of this concrete algebra in $B(\mathcal{H})$. Thus the constructed example is unital, even though unitality is not required by the existence statement.

Proof route

Use the constructed algebra T and its inclusion representation ρ_0. The classification theorem gives the single unitary-equivalence class, the compact-operator exclusion applies to ρ_0, and the density theorem gives the required value $\mathfrak{c}$.

Proof steps

1. Take the algebra T, its selected GNS direct sum $\mathcal{H}$, and the inclusion ρ_0. The representation-classification theorem says that ρ_0 is nonzero irreducible and that, for any nonzero irreducible $*$-representation $\rho : T \to B(H)$, a unitary $U : \mathcal{H} \to H$ satisfies $U\rho_0(t) = \rho(t)U$. This statement does not require $\rho(1) = I_H$ as an initial assumption.

2. The compact-operator exclusion rules out every injective representation of T onto the full algebra of compact operators, in particular the specified inclusion ρ_0. The density theorem gives $\mathrm{dens}_{\|\cdot\|}(T) = \mathfrak{c}$. These are precisely the other conditions of the existential predicate in the Lean statement.

Main citations

- The stated existence or structural result · Exact source
- The exact counterexample and density conditions · Exact source
- Unitary equivalence without an initial identity-preservation assumption · Exact source
- The same equivalence-class condition for the nonunital-algebra API · Exact source
- Exclusion of an injective representation onto the compact operators · Exact source
- Exact norm density of the constructed algebra · Exact source

Lean source signature (exact)

```
theorem existsNaimarkCounterexampleOfDensity_continuum :
    ExistsNaimarkCounterexampleOfDensity (Cardinal.continuum : Cardinal.{0})
```

> `(Cardinal.continuum : Cardinal.{0})` denotes the continuum $\mathfrak{c} = 2^{\aleph_0}$, explicitly typed as a cardinal of types in `Type 0` (also written `Type`). The colon is a type annotation, and `.{0}` specifies a universe level: it is not the cardinal number zero and imposes no countability assumption. The cardinal universe convention explains this type. The separately linked `ExistsNaimarkCounterexampleOfDensity` quantifies over the algebra, Hilbert space and representations in that universe. In the linked proof, `AtomicCounterexampleAlgebra` is T, `SelectedAtomicHilbert completedRootPureState` is $\mathcal{H}$, and `.toNonUnitalStarAlgHom` retains the same inclusion ρ_0 while forgetting the requirement to preserve the identity. `inferInstance` supplies the existing C*-algebra, Hilbert-space and spectral-order structures; these are not extra hypotheses of the theorem.

Lean realization notes

The fixed example also admits a faithful representation on a separable Hilbert space, but that is not a condition in this existence statement and is not asserted for every C*-algebra satisfying it.

▼ **Exact Card identity**

Language: en
CARD_CONTENT_COMPLETE
Card UID: lfh:lfh-declaration-card:sha256:9e4fa34878df2ba18cbc8247c4d9da4877ccd5abd32747c76de2e936615fa461
Card revision: 2 · SHA-256: 228ba19d3e6f019d7447205532137a8eabdb20491d59bbae4fa507ce55cb654f
Exposition revision: 3 · SHA-256: f9aed14a1af813b49cd2d48d81057a30890a81ebf2926060f799249cafc0e381
Source commit: 437e6e46228bbb8e91211ebded349d7a30020e73
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