Linked normal kernels share a projective coordinate

`MathlibAnnex.Analysis.CStarAlgebra.linked_normalPartPoint_iff_span_eq_or_span_eq`

theorem

Characterizes linked normal kernels by agreement of one projective coordinate.

Statement

Let H be a complex Hilbert space, and let $A \subseteq \mathcal{B}(H)$ be a nonzero norm-closed complex $*$-subalgebra acting irreducibly on H. An identity is not required. For $x, y \in H \setminus \{0\}$, define the bounded linear functional

$$f_{x,y} : A \to \mathbb{C}, \qquad f_{x,y}(a) = \langle y, a(x) \rangle.$$

The inner product is conjugate-linear in its first variable. The normal part is $N(A) = \{\ker f_{x,y} : x, y \in H \setminus \{0\}\}$. For $D \subseteq N(A)$, its inherited closure is

$$c(D) = \left\{ I \in N(A) : \bigcap_{J \in D} J \subseteq I \right\}.$$

Two points are **linked** when they are equal or their two-point set is not closed for this closure. For $x_1, x_2, y_1, y_2 \in H \setminus \{0\}$,

$$\ker f_{x_1,y_1} \text{ and } \ker f_{x_2,y_2} \text{ are linked} \iff \mathbb{C}x_1 = \mathbb{C}x_2 \text{ or } \mathbb{C}y_1 = \mathbb{C}y_2. \quad (1)$$

Assumptions

All four vectors are nonzero, but no unit-length or linear-independence assumption is imposed. Under these hypotheses, the coefficient-kernel membership theorem guarantees that every nonzero pair defines a point of the normal part. The closure in (1) is restricted to $N(A)$, not to arbitrary points of the ambient geometric structure space.

Conclusion

One shared projective coordinate characterizes a linked pair. In contrast, the coefficient-kernel equality theorem identifies equality of the two points with agreement of **both** projective coordinates. Linkedness here is the original pair relation, not its finite-chain transitive closure.

Main citations

- Linked-pair theorem: the equivalence (1), for the inherited normal-part closure.
- arXiv:2504.05551v1, Lemma 6.10: the linkedness criterion in the normal part.

Lean source signature (exact)

The declaration prefix below is a contiguous source excerpt, ending before its proof assignment.

```
theorem linked_normalPartPoint_iff_span_eq_or_span_eq
    (x₁ x₂ y₁ y₂ : H) (hx₁ : x₁ ≠ 0) (hx₂ : x₂ ≠ 0)
    (hy₁ : y₁ ≠ 0) (hy₂ : y₂ ≠ 0) :
    ((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S)).Linked
      (normalPartPoint S hS x₁ y₁ hx₁ hy₁) (normalPartPoint S hS x₂ y₂ hx₂ hy₂) ↔
      Submodule.span ℂ {x₁} = Submodule.span ℂ {x₂} ∨
      Submodule.span ℂ {y₁} = Submodule.span ℂ {y₂}
```

In the source, `S` denotes the algebra A used here.

In the source	Mathematical meaning

Hilbert space and irreducible algebra
`variable {H : Type u} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]`
`variable (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H))`
`variable [IsClosed (S : Set (H →L[ℂ] H))]`
`variable (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype S))`
	The complex Hilbert space H and the nonzero norm-closed complex $*$-subalgebra $A \subseteq \mathcal{B}(H)$ acting irreducibly. The source calls the algebra `S`.

Four nonzero vectors `(x₁ x₂ y₁ y₂ : H) (hx₁ : x₁ ≠ 0) (hx₂ : x₂ ≠ 0) (hy₁ : y₁ ≠ 0) (hy₂ : y₂ ≠ 0) :`	The vectors x_1, x_2, y_1, y_2 belong to $H \setminus \{0\}$. There is no normalization or independence condition.
Linked coefficient kernels `((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S)).Linked (normalPartPoint S hS x₁ y₁ hx₁ hy₁) (normalPartPoint S hS x₂ y₂ hx₂ hy₂) ↔`	The two points are $\ker f_{x_1,y_1}$ and $\ker f_{x_2,y_2}$ in $N(A)$. The outer `subspace` gives the relative closure c on $N(A)$. Linkedness means that the points are equal or that their two-point set is not closed; it does not mean a finite chain of links.
Projective source or target equality `Submodule.span ℂ {x₁} = Submodule.span ℂ {x₂} ∨ Submodule.span ℂ {y₁} = Submodule.span ℂ {y₂} := by`	The two alternatives in (1) are $$\mathbb{C}x_1 = \mathbb{C}x_2 \quad \text{or} \quad \mathbb{C}y_1 = \mathbb{C}y_2.$$ Each span is the complex line generated by its nonzero vector. The disjunction classifies linkedness; equality of the two kernels requires both line equalities, as stated in the separately cited result.

Linkedness includes equal points. The theorem concerns one linked pair, not a finite chain.

Exact surrounding source context (separate excerpts)

Exact surrounding source:

```
variable {H : Type u} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
variable (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H))
variable [IsClosed (S : Set (H →L[ℂ] H))]
variable (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype
S))
```

Related exact source for the reading guide

Projective source or target equality · Fixed source

```
      Submodule.span ℂ {x₁} = Submodule.span ℂ {x₂} v
      Submodule.span ℂ {y₁} = Submodule.span ℂ {y₂} := by
```