No nonzero irreducible representation on a separable Hilbert space

`MathlibAnnex.CStarAlgebra.CAR.not_isIrreducible_of_separable`

Distinguishes separable representability from separable irreducible representability.

Statement

The fixed CAR-based C*-algebra A has no nonzero irreducible representation on any separable complex Hilbert space.

Assumptions

Let $A \subseteq B(H_{at})$ be the fixed unital C*-algebra obtained by adjoining the chosen shell-link unitaries to the atomic representation of the CAR algebra C, and then taking the norm-closed unital *-algebra they generate. Let K be any separable complex Hilbert space, and let $\sigma: A \to B(K)$ be any complex-linear multiplicative *-preserving map. It need not preserve the unit, and no faithfulness is assumed.

Conclusion

σ is not both nonzero and irreducible. Equivalently, no such K and σ give a nonzero irreducible representation of A.

Proof route

If σ were nonzero and irreducible, it would be unital. The uniqueness of the atomic irreducible model would make it a separable singleton model, so the proved Rosenberg consequence would force A to be finite-dimensional, a contradiction.

Proof steps

1. Convert the assumed nonzero irreducible comparison map to a unital representation.

2. Compare it and all other irreducible representations through the atomic model, obtaining the required singleton property.

3. Apply the separable singleton finite-dimensionality theorem and contradict the infinite-dimensionality of A established by the counterexample theorem for A.

Lean realization notes

The source predicate IsIrreducible includes nonzeroness, so the zero map is no counterexample to this statement. K ranges over an arbitrary independent universe v. This is not a claim that A is nonprimitive: its faithful atomic irreducible representation remains available. The comma in ∀ σ, ¬ σ.IsIrreducible separates the quantified variable from the proposition; it is not a substitute for an implication arrow.

Language: en · CARD_CONTENT_COMPLETE

Main citations

MathlibAnnex.CStarAlgebra.CAR.atomicCounterexampleEndpoint

MathlibAnnex.Analysis.CStarAlgebra.Representation.finiteDimensional_algebra_of_singleton_amongNonUnital

Lean source declaration (exact)

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/-- The main target admits no nonzero irreducible ordinary representation on
a separable complete complex Hilbert space. The input representation is not
assumed to preserve the unit. -/
theorem not_isIrreducible_of_separable
{H : Type v} [NormedAddCommGroup H] [InnerProductSpace ℂ H]
[CompleteSpace H] [TopologicalSpace.SeparableSpace H]
(rho : NonUnitalRepresentation (A := AtomicCounterexampleAlgebra) (H := H)) :
¬ rho.IsIrreducible := by
intro hrho
let pi : Representation AtomicCounterexampleAlgebra H := rho.toUnital hrho
have hpi : Representation.IsIrreducible pi :=
NonUnitalRepresentation.isIrreducible_toUnital rho hrho
have hambient_pi : atomicCounterexampleRepresentation.UnitaryEquivalent pi := by
obtain ⟨U, hU⟩ := (atomicCounterexampleEndpoint.{v}).captures_nonunital H rho hrho
refine ⟨U, ?_⟩
intro a x
simpa [atomicCounterexampleRepresentation, pi] using hU a x
have hsingleton :
Representation.IsSingletonIrreducibleModelAmongNonUnital.{0, v, 0} pi := by
refine ⟨hpi, ?_⟩
intro K _ _ _ sigma hsigma
have hambient_sigma :
atomicCounterexampleRepresentation.UnitaryEquivalent (sigma.toUnital hsigma) := by
obtain ⟨U, hU⟩ := (atomicCounterexampleEndpoint.{0}).captures_nonunital K sigma hsigma
refine ⟨U, ?_⟩
intro a x
change U (atomicCounterexampleRepresentation a x) = sigma a (U x)
simpa [atomicCounterexampleRepresentation] using hU a x
exact Representation.unitaryEquivalent_trans
(Representation.unitaryEquivalent_symm hambient_pi) hambient_sigma
exact (atomicCounterexampleEndpoint.{v}).not_finiteDimensional_target
(Representation.finiteDimensional_algebra_of_singleton_amongNonUnital
pi hsingleton)
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Read exact MathlibAnnex v0.4.0 source

Exact Card identity

Stable Card ID: a3bcf63be1be2f724252fe0b1f11393935aafaea9c2f2fa70189d23c11bb0f65

Card SHA-256: ad091ef1bdb46599561192f9b2e32004e2aa24220ab5bf053514f7acb52b56d2

Approved exposition SHA-256: 7cae6ae85cfa8d9830c5ec577b007152920d0fc6cce306be0afb5060d09ba7df

AI-assisted material. Selected, edited, and reviewed by Ryotaro Tanaka. The recorded review is source–exposition correspondence, not an independent proof audit.