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No nonzero irreducible representation on a separable Hilbert space

`MathlibAnnex.CStarAlgebra.CAR.not_isIrreducible_of_separable`

theorem

Rules out irreducible representations of T on separable Hilbert spaces by the finite-dimensionality theorem for a single irreducible representation class.

Statement

Let A be the completed CAR algebra. Let $\pi : A \to B(\mathcal{H})$ be the direct sum of one chosen GNS representation from each pure-state GNS equivalence class, and let $(L_i)_{i \in I}$ be the unitaries obtained from the fixed CAR homogeneity and shell construction. Put $T = C^*(\pi(A) \cup \{L_i : i \in I\}) \subseteq B(\mathcal{H})$, and write $\iota(a) = \pi(a)$ for the embedding into T. For every separable complex Hilbert space H, there is no nonzero irreducible $*$-representation

$$\rho : T \longrightarrow B(H).$$

Preservation of the identity is not assumed. Here irreducibility means that the only closed subspaces of H invariant under every $\rho(t)$ are $\{0\}$ and H; for a $*$-representation these subspaces also reduce the representation.

Assumptions

The algebra T is the particular algebra constructed above, not an arbitrary C*-algebra. Its established properties are infinite dimensionality and a faithful irreducible inclusion $\rho_0 : T \to B(\mathcal{H})$ to which every nonzero irreducible representation is unitarily equivalent. Separability is required of H, not of T; faithfulness and preservation of the identity are not assumptions on ρ.

Conclusion

Every nonzero $*$-representation of T on a separable Hilbert space is reducible. This does not rule out irreducible representations on nonseparable spaces: the inclusion ρ_0 is one.

Proof route

Suppose that a nonzero irreducible representation ρ exists on a separable H. It must preserve the identity. Composing its unitary equivalence with ρ_0 with the equivalence for an arbitrary irreducible representation shows that ρ represents the unique irreducible representation class. The cited finite-dimensionality theorem then contradicts the infinite dimensionality of T.

Proof steps

1. Put $P = \rho(1)$. The $*$-representation identities give $P^* = P$, $P^2 = P$, and $P\rho(t) = \rho(t)P = \rho(t)$. Thus P is an orthogonal projection with reducing range. Since $\rho \neq 0$, one has $P \neq 0$; irreducibility therefore forces $P = I_H$. The same representation ρ is consequently unital; no new map is introduced.

2. The classification theorem gives a unitary $U_\rho : \mathcal{H} \to H$ with $U_\rho \rho_0(t) = \rho(t) U_\rho$. For any nonzero irreducible $*$-representation $\sigma : T \to B(K)$, the same theorem gives a unitary $U_\sigma : \mathcal{H} \to K$ with $U_\sigma \rho_0(t) = \sigma(t) U_\sigma$. There is no separability assumption on K.

3. Set $V = U_\sigma U_\rho^{-1} : H \to K$. For every $t \in T$,

   $$\begin{aligned} V\rho(t) &= U_\sigma \rho_0(t) U_\rho^{-1} \\ &= \sigma(t) U_\sigma U_\rho^{-1} = \sigma(t) V. \end{aligned}$$

   Hence every nonzero irreducible representation σ is unitarily equivalent to ρ.

4. The cited finite-dimensionality theorem applies to a nonzero unital C*-algebra with exactly one unitary-equivalence class of nonzero irreducible representations, when that class has a representative on a separable Hilbert space. The representation ρ on H satisfies precisely these hypotheses, so $\dim_{\mathbb{C}} T < \infty$. But ι embeds the infinite-dimensional CAR algebra into T. This is the contradiction.

Main citations

- The stated existence or structural result · Exact source
- Unitary equivalence of all nonzero irreducible representations · Exact source
- Unit preservation forced by nonzero irreducibility · Exact source
- Unital rebundling with unchanged operators · Exact source
- A single unitary-equivalence class of nonzero irreducible representations · Exact source
- Finite dimensionality when the unique irreducible class has a separable representative · Exact source

Lean source signature (exact)

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theorem not_isIrreducible_of_separable
    {H : Type v} [NormedAddCommGroup H] [InnerProductSpace ℂ H]
    [CompleteSpace H] [TopologicalSpace.SeparableSpace H]
    (rho : NonUnitalRepresentation (A := AtomicCounterexampleAlgebra) (H := H)) :
    ¬ rho.IsIrreducible
```

> `NonUnitalRepresentation` means a $*$-representation with no requirement that `rho 1 = 1`; it does not mean that the represented algebra lacks an identity. `IsIrreducible` here includes nonzeroness. The local `pi := rho.toUnital hrho` in the linked proof is the same ρ after $\rho(1) = I_H$ has been proved, not the direct-sum representation π of CAR. `hambient_pi` and `hambient_sigma` give U_ρ and U_σ; `hsingleton` states that every nonzero irreducible representation is unitarily equivalent to ρ. This is the hypothesis of `finiteDimensional_algebra_of_singleton_amongNonUnital`. The parameter v allows the Hilbert space H to lie in an independent universe; the comparison spaces in that finite-dimensionality theorem lie in the universe of T.

Lean realization notes

The theorem places no separability restriction on the Hilbert space of another irreducible representation used for comparison. It asserts nonexistence only for the proposed representation on H.

▼ **Exact Card identity**

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