An irreducible target representation has a surviving fixed space

`MathlibAnnex.CStarAlgebra.CAR.exists_nonzero_targetFixedSpace`

theorem

Rules out simultaneous disappearance of all limiting CAR flags by passing reduction through the reconstructed generators.

Statement

For the shell family and generated algebra described below, let $\rho : T \to B(K)$ be a nonzero irreducible unital star representation and put $\sigma = \rho \circ \iota$. Then there is an index $i \in I$ such that $F_i = \bigcap_{n \ge 0} \operatorname{ran} \sigma(f_{i,n})$ is nonzero.

Assumptions

Let A be the completed CAR algebra, obtained by completing the matrix stages $M_{2^n}(\mathbb{C})$ with embeddings $a \mapsto a \otimes I_2$. Let f_n be the image of the first diagonal matrix unit. Thus $f_0 = 1$ and the f_n form a decreasing sequence of projections. The root state ω has value a_{00} on a matrix a in any stage.

Choose a pure state φ_j in each unitary-equivalence class $j \in I$ of pure-state GNS representations, with $\varphi_o = \omega$ at the root index o. Denote its complete complex GNS space, unital star representation and unit cyclic vector by (H_j, π_j, ξ_j). Set $\mathcal{H} = \bigoplus_{j \in I} H_j$ and $\pi = \bigoplus_{j \in I} \pi_j$. The index set need not be countable.

A supplied shell family consists of unital complex star automorphisms α_j with $\varphi_j \circ \alpha_j = \omega$ and elements $w_{j,n} \in A$. Put $f_{j,n} = \alpha_j(f_n)$. Their support identities are $w_{j,n}^* w_{j,n} = f_{j,n} - f_{j,n+1}$ and $w_{j,n} w_{j,n}^* = f_n - f_{n+1}$. At the root, α_o is the identity and $w_{o,n} = f_n - f_{n+1}$.

Let L_j be unitaries on $\mathcal{H}$ satisfying $L_j \pi(f_{j,n} - f_{j,n+1}) = \pi(w_{j,n})$ for every j, n. Let $T \subseteq B(\mathcal{H})$ be the norm-closed unital star algebra generated by $\pi(A)$ and all L_j. Write $\iota : A \to T$ for $\iota(a) = \pi(a)$ and $g_j \in T$ for the element represented by L_j. For a unital complex star representation $\rho : T \to B(K)$ on a complete complex Hilbert space, put $\sigma = \rho \circ \iota$ and $V_j = \rho(g_j)$. Each V_j is unitary because ρ preserves the unit, products and adjoints. Here irreducibility includes nonzero representation and means that the only closed subspaces invariant under every represented operator and its adjoint are $\{0\}$ and K. No root normalization $L_o = I$ or prescribed transport of the atomic cyclic vectors is assumed.

Conclusion

At least one transported flag retains a nonzero common fixed vector. This supplies the initial vector for the residual-corner construction of vector states. The assertion concerns existence of some i; it neither chooses a distinguished index nor says that this common range is one-dimensional.

Proof route

Suppose all F_i vanish, and write P_i for the orthogonal projection onto F_i. Let S_i be the strong sum of the represented shell links. By the shell reconstruction theorem, in each reconstructed generator $V_i = S_i + R_i$, the initial residual projection is zero, so $R_i = 0$. Every closed reducing subspace for σ therefore reduces every V_i, and hence the entire target representation. Thus σ would be irreducible. Coverage by selected pure-state GNS representations then transports a selected cyclic vector back to a nonzero vector fixed by one of the flags, contradicting the assumption.

Proof steps

1. Let $M \subseteq K$ be a closed subspace reducing σ. Every $\sigma(w_{i,n})$ and its adjoint preserve M, hence so do their finite sums. Closure of M passes this invariance to the two strong limits S_i and S_i^*. Under $F_i = \{0\}$, the formula $R_i = V_i P_i$ gives $R_i = 0$, so M reduces V_i.

2. The orthogonal projection onto M commutes with the represented source and all added generators. Its commutation relation persists under algebraic star operations and norm limits. Consequently M reduces every $\rho(t)$, $t \in T$, and irreducibility of ρ forces $M = \{0\}$ or K. The space K is nonzero and σ is unital, so this proves nonzero irreducibility of σ under the supposition.

3. Coverage by selected pure-state GNS representations supplies an index j and a unitary $e : K \to H_j$ with $e\sigma(a) = \pi_j(a)e$ for every $a \in A$. The vector $\eta = e^{-1}\xi_j$ has norm one. Since $\varphi_j(f_{j,n}) = \omega(f_n) = 1$, each projection $\pi_j(f_{j,n})$ fixes ξ_j; intertwining therefore gives $\sigma(f_{j,n})\eta = \eta$ for all n.

4. A fixed vector of a projection belongs to its range. Thus $\eta \in F_j$, contradicting $F_j = \{0\}$ and $\|\eta\| = 1$. This completes the exclusion of the all-zero case.

Main citations

- Exact declaration and proof · Exact source
- MathlibAnnex.CStarAlgebra.CAR.RepresentativeShellFamily · Exact source
- MathlibAnnex.CStarAlgebra.CAR.exists_targetShellReconstruction · Exact source
- MathlibAnnex.CStarAlgebra.CAR.isIrreducible_restrictedRepresentation_of_all_fixed_bot · Exact source
- MathlibAnnex.CStarAlgebra.AtomicConstruction.reduces_concreteTarget_of_generators · Exact source
- MathlibAnnex.CStarAlgebra.irreducible_covered_by_pureState_representative · Exact source
- MathlibAnnex.CStarAlgebra.CAR.selectedVector_fixed_transportedFlag · Exact source
- The root state bundled with its purity proof · Exact source

Lean source signature (exact)

```
theorem exists_nonzero_targetFixedSpace
    (family : RepresentativeShellFamily)
    (L : MathlibAnnex.CStarAlgebra.PureState.GNSClass Limit →
      MathlibAnnex.CStarAlgebra.PureState.SelectedAtomicHilbert completedRootPureState →L[ℂ]
        MathlibAnnex.CStarAlgebra.PureState.SelectedAtomicHilbert completedRootPureState)
    (hLunit : ∀ i, L i ∈ unitary
      (MathlibAnnex.CStarAlgebra.PureState.SelectedAtomicHilbert completedRootPureState →L[ℂ]
        MathlibAnnex.CStarAlgebra.PureState.SelectedAtomicHilbert completedRootPureState))
    (hLsource : ∀ i n, (L i).comp (selectedAtomicRepresentation
      (transportedFlag family i n - transportedFlag family i (n + 1))) =
        representedShellLink family i n)
    {K : Type v} [NormedAddCommGroup K] [InnerProductSpace ℂ K]
    [CompleteSpace K] (rho : Representation (AtomicTarget L) K)
    (hrho : rho.IsIrreducible) :
    ∃ i : MathlibAnnex.CStarAlgebra.PureState.GNSClass Limit,
      (⨅ n, ((restrictedRepresentation L rho)
        (transportedFlag family i n)).range) ≠ ⊥
```

> Here `Limit` is A, `completedRootPureState` is the root state ω bundled with its purity proof, and `SelectedAtomicHilbert completedRootPureState` is the atomic Hilbert space $\mathcal{H}$. `hrho : rho.IsIrreducible` includes nonzero irreducibility of ρ. The expression `⊓ n, ... .range` is F_i, and `≠ ⊥` means that this subspace is not $\{0\}$. `restrictedRepresentation L rho` is σ; the source conclusion quantifies over an index `i` and does not claim irreducibility of σ without the temporary all-zero hypothesis used in the proof.

Lean realization notes

The temporary assertion that σ is irreducible occurs inside the contradiction argument under the all-zero hypothesis. It is not asserted for every source restriction. Strong limits are taken on K, while extension from source and generators to T uses norm-closed generation.

▼ **Exact Card identity**

Language: en

CARD_CONTENT_COMPLETE

Card UID: lfh:lfh-declaration-card:sha256:a78c4de54da4fbe1ab39f1d488112128a74cc98680c7764d29e113cb74ba0efc

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Exposition revision: 1 · SHA-256: 17a142f497c21610256d4f3f2cb3df7b8d046f032b3ee22b748339048333cb66

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