Unique extension of the CAR trace among all states

`MathlibAnnex.CStarAlgebra.CAR.existsUnique_state_extension_trace`

Establishes uniqueness of a state extension without assuming the extension is tracial.

Statement

For every representative shell family F, the normalized trace τC on the CAR algebra C extends to exactly one state of the associated C*-algebra A_F.

Assumptions

Let C be the completed CAR algebra, φ_0 its distinguished pure state, φ_i one selected pure state in each GNS-equivalence class, and e_n the fixed root shell projections. Fix a representative shell family F: for each i, an automorphism α_i and elements $w_{i,n}$ of C satisfying $\varphi_i \circ \alpha_i = \varphi_0$, $w_{i,n}^* w_{i,n} = \alpha_i(e_n)$, and $w_{i,n} w_{i,n}^* = e_n$. At the root class, α_0 is the identity and $w_{0,n} = e_n$. Let $A\_F \subseteq B(H_{at})$ be the norm-closed unital *-algebra generated by the atomic image of C and the shell-link unitaries constructed from F; write $j\_F: C \to A\_F$ for the source map. Let τC be the normalized CAR trace. The family parameter F is retained; it is not replaced by an arbitrary C*-algebra.

Conclusion

There exists a unique continuous complex-linear functional $\varphi: A\_F \to \mathbb{C}$ which is positive, satisfies $\varphi(1) = 1$, and obeys $\varphi(j\_F(c)) = \tau C(c)$ for every $c \in C$.

Proof route

The constructed trace extension gives existence. Compare the cyclic GNS representation of any other state extension with the constructed tracial representation by a pointed unitary; equality of their vector states gives uniqueness.

Proof steps

1. Use the known extension to obtain a normalized positive functional with the required restriction.
2. Apply the pointed-unitary comparison theorem to an arbitrary state extension and its cyclic GNS vector.
3. Evaluate the equal vector functionals on each $a \in A_F$.

Lean realization notes

The source theorem is family-parametric. Uniqueness is among all state extensions of this particular trace, not merely among tracial extensions, and not among all states on A_F. Traciality and faithfulness are separate conclusions elsewhere. Taking the fixed homogeneity family recovers the algebra A.

Language: en · CARD_CONTENT_COMPLETE

Main citations

MathlibAnnex.CStarAlgebra.CAR.eq_traceExtension_of_mem_stateSpace_of_apply_shellFamilySourceHom_eq_trace

Lean source declaration (exact)

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/-- Existence and uniqueness are on the actual fixed target. -/
theorem existsUnique_state_extension_trace (family : RepresentativeShellFamily) :
∃! φ : ShellFamilyTarget family →L[ℂ] ℂ,
φ ∈ MathlibAnnex.Analysis.CStarAlgebra.stateSpace (ShellFamilyTarget family) ∧
∀ b, φ (shellFamilySourceHom family b) = trace b := by
refine ⟨traceExtension family,
⟨traceExtension_mem_stateSpace family, traceExtension_shellFamilySourceHom family⟩, ?_⟩
intro φ hφ
exact eq_traceExtension_of_mem_stateSpace_of_apply_shellFamilySourceHom_eq_trace family φ hφ.1
hφ.2
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Read exact MathlibAnnex v0.4.0 source

Exact Card identity

Stable Card ID: a7a18351b64a5f91df84a76896ea51041aecb8df538a15ac9cb54e956ccbac27

Card SHA-256: 4e50c44f14a48c8f061646a405c75ba136829cab84f8fd45d4e52cf87386ca33

Approved exposition SHA-256: 382d7ea8db14b50d73afd82b3ccadbf192dc8b3d88e57d7b80492f4cea2e70f2

AI-assisted material. Selected, edited, and reviewed by Ryotaro Tanaka. The recorded review is source–exposition correspondence, not an independent proof audit.