Uniqueness among all state extensions of the CAR trace

`MathlibAnnex.CStarAlgebra.CAR.existsUnique_state_extension_trace`

theorem

Pointed transport identifies arbitrary state extensions before traciality is proved.

Statement

Let A be the completed CAR algebra and τ its normalized trace. Fix a representative shell family and its chosen unitary links L_j in the selected atomic representation π. Write $T = C^*(\pi(A), \{L_j\})$ for the resulting concrete target and $\iota : A \to T$, $\iota(a) = \pi(a)$, for its faithful unital source map. This fixes one target and one source map throughout. There is exactly one state ϕ on T such that $\phi \circ \iota = \tau$. Uniqueness here ranges over all state extensions, including those not assumed tracial.

Assumptions

Only the fixed representative shell family is given. A competing functional ψ must be a state on T and satisfy $\psi(\iota(a)) = \tau(a)$ for every $a \in A$.

Conclusion

Every such ψ equals the chosen extension ϕ. The existence and uniqueness refer to the same fixed algebra T.

Proof route

Construct the two target GNS representations and apply pointed unitary transport from their common source trace. Equality of the vector functionals then gives equality of states.

Proof steps

1. Existence is `exists_state_extension_trace`. For an arbitrary competing state ψ, let $(H_\psi, \rho_\psi, \xi_\psi)$ be its GNS representation; define $(H_\phi, \rho_\phi, \xi_\phi)$ for the chosen extension. Both target orbits are dense and their source vector functionals are τ.

2. Apply `exists_pointed_unitary_of_trace_of_cyclic` with the ψ representation first and the ϕ representation second. Its hypotheses are exactly the two restriction equalities and target cyclicity. It gives a unitary

$$E_\psi : H_\psi \longrightarrow H_\phi, \quad E_\psi \xi_\psi = \xi_\phi, \quad E_\psi \rho_\psi(t) = \rho_\phi(t) E_\psi \quad (t \in T).$$

3. For every $t \in T$, preservation of the inner product and these two identities give

$$\begin{aligned} \psi(t) &= \langle \xi_\psi, \rho_\psi(t)\xi_\psi \rangle \\ &= \langle E_\psi \xi_\psi, E_\psi \rho_\psi(t)\xi_\psi \rangle \\ &= \langle \xi_\phi, \rho_\phi(t)\xi_\phi \rangle = \phi(t). \end{aligned}$$

This is the pointwise equality used by `eq_traceExtension_of_mem_stateSpace_of_apply_shellFamilySourceHom_eq_trace`; extensionality then gives $\psi = \phi$.

Main citations

- The stated existence or structural result · Exact source
- The fixed concrete target and source map · Exact source
- Faithfulness of the source embedding · Exact source
- Existence of a state extension · Exact source
- Pointed transport for trace-realizing cyclic targets · Exact source
- Equality of an arbitrary extension with the chosen state · Exact source

Lean source signature (exact)

```
theorem existsUnique_state_extension_trace (family : RepresentativeShellFamily) :
    ∃! ϕ : ShellFamilyTarget family →L[ℂ] ℂ,
      ϕ ∈ MathlibAnnex.Analysis.CStarAlgebra.stateSpace (ShellFamilyTarget family) ∧
        ∀ b, ϕ (shellFamilySourceHom family b) = trace b
```

> `∃! ϕ` quantifies a continuous functional on the fixed `ShellFamilyTarget family`. The conjunction requires state-space membership and restriction equal to `trace`. The linked uniqueness supplier calls the arbitrary competing state `ϕ` and its positive-map bundle `f`; these correspond to ψ and its positive bundle in the mathematics above. Its proof-local `ρ`, `ξ`, and `e` are the competitor's GNS action, vector and pointed transport toward the chosen extension. Thus E_ψ records the direction of that `e`.

Lean realization notes

The argument does not use uniqueness among tracial states to prove this stronger extension uniqueness. The unitary below compares the GNS representations of two possibly different extension states; it is not an identification of independently chosen vectors without a transport theorem.

▼ **Exact Card identity**

Language: en
CARD_CONTENT_COMPLETE
Card UID: lfh:lfh-declaration-card:sha256:a7a18351b64a5f91df84a76896ea51041aecb8df538a15ac9cb54e956ccbac27
Card revision: 2 · SHA-256: 4e50c44f14a48c8f061646a405c75ba136829cab84f8fd45d4e52cf87386ca33
Exposition revision: 3 · SHA-256: e14c11c65ccfaa24799d351ac5da4d424bf2f2045b087d272589498455e3306c
Source commit: 437e6e46228bbb8e91211ebded349d7a30020e73
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