Irreducible dimension equal to a fixed number

`MathlibAnnex.Analysis.CStarAlgebra.forall_irreducible_finrank_eq_iff_component_compact_homeomorph`

theorem

Characterizes one fixed irreducible dimension by finite-dimensional compact component models.

Statement

Let A be a nonzero complex C*-algebra, and let $n \in \mathbb{N}$.

For a complex C*-algebra B, write $S(B)$ for its geometric structure space. Its points are the unions of kernels of norm-one bounded complex-linear functionals supporting a maximal proper face of the closed unit ball of B; faces use real convexity. Write c_B for the hull-kernel closure on $S(B)$:

$$c_B(\mathcal{U}) = \left\{ J \in S(B) : \bigcap_{L \in \mathcal{U}} L \subseteq J \right\}, \qquad \mathcal{U} \subseteq S(B).$$

These are the geometric structure and its closure, with the norm-one description justified by the support-functional norm theorem.

For $I \in S(A)$, let $C_A(I)$ be its finite-chain component: the points reachable from I by a finite chain in which adjacent points are equal or form a nonclosed two-point set. This uses the linked-pair relation and its generated components, rather than topological connectedness. Give $C_A(I)$ the induced closure

$$c_{C_A(I)}(E) = C_A(I) \cap c_A(E), \qquad E \subseteq C_A(I).$$

For a complex Hilbert space H, let $\mathcal{K}(H)$ be the algebra of compact operators on H. In particular, $c_{\mathcal{K}(K)}$ denotes closure on $S(\mathcal{K}(K))$, where $\mathcal{K}(K)$ is itself the algebra to which the geometric structure construction is applied.

The following conditions are equivalent:

- For every irreducible $*$-representation $\pi : A \to \mathcal{B}(H)$ on a complex Hilbert space H,

$$H \text{ is finite-dimensional} \quad \text{and} \quad \dim_{\mathbb{C}} H = n. \qquad (1)$$

- For every $I \in S(A)$, there are a nonzero finite-dimensional complex Hilbert space K with $\dim_{\mathbb{C}} K = n$ and a bijection $e : C_A(I) \to S(\mathcal{K}(K))$ satisfying

$$e(c_{C_A(I)}(E)) = c_{\mathcal{K}(K)}(e(E)) \quad (E \subseteq C_A(I)). \qquad (2)$$

The same model K has dimension n and supports the homeomorphism e. The number n is fixed for all components, while K and e may depend on I. Equality (2) expresses continuity of the same e and e^{-1} for arbitrary subsets; $e(E)$ denotes the image of a set.

Assumptions

The algebra A is nonzero; no assumption $n > 0$ is made. Representations are $*$-homomorphisms into bounded operators, without a requirement of unitality. Irreducibility includes a nonzero action, but does not require faithfulness. All Hilbert spaces are complex and complete. Finite-dimensionality is a conclusion on the representation side and a requirement on the model side. It is separate from the numerical dimension equation. No separability hypothesis is added.

Conclusion

Every irreducible representation space is finite-dimensional of the same dimension n exactly when each geometric component has a compact-operator model of that dimension. For nonzero A, both conditions fail at $n = 0$, as specified by the zero-dimension exclusion and the equivalence above. This does not add positivity of n as a hypothesis.

Main citations

The fixed-dimension characterization specializes the general dimension-condition theorem to equality with n. The finite-dimensional compact-model condition retains both finiteness and the numerical condition on the same target. The same-map upgrade theorem supplies full continuity without replacing the finite comparison or its inverse. The independent-universe dimension transport and the exclusion of zero dimension specify those two scope points.

The finite-homeomorphism fields, closure continuity, and homeomorphism fields specify the map conditions used here. The closure-transport identity expresses a homeomorphism by equality of closures of arbitrary sets.

Lean source signature (exact)

The preceding `variable` declaration introduces A and its nontriviality. The local spectral order is used for positivity and pure states. The `include hA in` retains nonzero A among this theorem's parameters, although the main excerpt does not repeat these ambient binders. Both separate excerpts are shown in the source context below.

HomogeneousCharacterization.lean. The contiguous declaration prefix ends before the proof assignment.

```
theorem forall_irreducible_finrank_eq_iff_component_compact_homeomorph (n : ℕ) :
    (∀ (L : Type u) [NormedAddCommGroup L] [InnerProductSpace ℂ L] [CompleteSpace L],
      ∀ pi : NonUnitalCStarRepresentation A L, pi.IsIrreducible →
        FiniteDimensional ℂ L ∧ Module.finrank ℂ L = n) ↔
    ∀ I : NormedSpace.GeometricStructureSpace ℂ A,
      ClosureSpace.HasFiniteDimensionalCompactHomeomorph.{u, u}
        ((NormedSpace.geometricStructureClosure ℂ A).subspace
          ((NormedSpace.geometricStructureClosure ℂ A).component I)) (fun m => m = n)
```

The component condition uses the following separate exact definition, including its complete right-hand side: HomogeneousCharacterization.lean.

```
def HasFiniteDimensionalCompactHomeomorph.{uH, uX} {X : Type uX}
    (c : ClosureSpace X) (P : ℕ → Prop) : Prop :=
  ∃ (H : Type uH) (instN : NormedAddCommGroup H),
    letI := instN
    ∃ instI : InnerProductSpace ℂ H,
      letI := instI
      ∃ instC : CompleteSpace H,
        letI := instC
        ∃ instT : Nontrivial H,
          letI := instT
          FiniteDimensional ℂ H ∧ P (Module.finrank ℂ H) ∧
            Nonempty (Homeomorph c
              (NormedSpace.geometricStructureClosure ℂ
                (ContinuousLinearMap.compactOperatorStarSubalgebra ℂ H)))
```

In the source	Mathematical meaning

Exact declaration
Ambient inputs
`variable {A : Type u} [NonUnitalCStarAlgebra A] [hA : Nontrivial A]`
`attribute [local instance] _root_.CStarAlgebra.spectralOrder _root_.CStarAlgebra.spectralOrderedRing`
	The nonzero complex C*-algebra A. The two local instances give its compatible spectral order, used for positivity and pure states. They do not introduce another choice of order. The separate `include hA in` shown below retains nontriviality in this theorem; no unit is required.

Exact declaration Representation quantifiers `(∀ (L : Type u) [NormedAddCommGroup L] [InnerProductSpace ℂ L] [CompleteSpace L], ∀ pi : NonUnitalCStarRepresentation A L, pi.IsIrreducible →`	Every complex Hilbert space H and every irreducible $*$-representation $\pi : A \to \mathcal{B}(H)$. Here source `L` is the representation space called H in the Statement. Its norm, complex inner product and completeness are required by the three bracketed inputs. Irreducibility includes a nonzero action, not faithfulness.

In the source	Mathematical meaning
Exact declaration Representation condition `FiniteDimensional ℂ L ∧ Module.finrank ℂ L = n) ↔`	The representation space H is finite-dimensional and satisfies $\dim_{\mathbb{C}} H = n$, as in (1). Both clauses are required. The parameter `(n : ℕ)` in the main declaration fixes one natural number throughout the equivalence.
Exact declaration Component quantifier `∀ I : NormedSpace.GeometricStructureSpace ℂ A,`	Every point $I \in S(A)$. The model space and comparison map required after this quantifier may depend on I. The number n remains fixed.
Exact declaration Comparison applied to the component `ClosureSpace.HasFiniteDimensionalCompactHomeomorph. {u, u} ((NormedSpace.geometricStructureClosure ℂ A).subspace ((NormedSpace.geometricStructureClosure ℂ A).component I)) (fun m => m = n)`	Apply the displayed comparison definition to the closure induced on $C_A(I)$. `geometricStructureClosure ℂ A` is the closure c_A on $S(A)$; its `component I` is the point set $C_A(I)$. `subspace` makes that set a closure space, with $$c_{C_A(I)}(E) = C_A(I) \cap c_A(E)$$ for $E \subseteq C_A(I)$. This closure is the argument `c` of the definition above, and its underlying set is `X`. The final argument `(fun m => m = n)` assigns $P(m)$ the condition $m = n$ on the model dimension.
Exact declaration Model and inner product `∃ (H : Type uH) (instN : NormedAddCommGroup H), letI := instN ∃ instI : InnerProductSpace ℂ H, letI := instI`	The condition requires a model space K, called `H` in this definition, with the following structures. `instN` gives its normed additive group, and `instI` its complex inner product. Each following `letI` uses that choice on the same space, first to form `InnerProductSpace ℂ H`, then to interpret the remaining Hilbert-space conditions.
Exact declaration Completeness and nonzero model `∃ instC : CompleteSpace H, letI := instC ∃ instT : Nontrivial H, letI := instT`	`instC` requires this same K to be complete; `instT` requires $K \neq \{0\}$. Their `letI` declarations retain these structures when forming bounded operators, the compact-operator algebra and its geometric closure. They do not choose a second space.
Exact declaration Dimension of the model `FiniteDimensional ℂ H ∧ P (Module.finrank ℂ H) ∧`	This same model K must be finite-dimensional. Substituting $P(m) \equiv (m = n)$ gives $\dim_{\mathbb{C}} K = n$. The conjunction separately retains finite-dimensionality and the numerical equation. It does not replace them by the `finrank` equation alone.
Exact declaration Homeomorphism existence `Nonempty (Homeomorph c (NormedSpace.geometricStructureClosure ℂ (ContinuousLinearMap.compactOperatorStarSubalgebra ℂ H)))`	`Nonempty` requires an element of this `Homeomorph` type: one bijection $e : C_A(I) \to S(\mathcal{K}(K))$ with both continuity fields below. Its target closure is $c_{\mathcal{K}(K)}$ on the geometric structure of the compact-operator algebra on the same model K, not on K itself.

In the source	Mathematical meaning
Exact declaration Homeomorphism and forward field `structure Homeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where continuous_toFun : c.Continuous d toFun`	`extends α ≃ β` supplies the bijection, extracted as `e.toEquiv`. Its forward map `toFun` is e. In this application, $c = c_{C_A(I)}$ and $d = c_{\mathcal{K}(K)}$, so the first field says $$e(c_{C_A(I)}(E)) \subseteq c_{\mathcal{K}(K)}(e(E))$$ for every $E \subseteq C_A(I)$.
Exact declaration Inverse field of the same homeomorphism `continuous_invFun : d.Continuous c invFun`	`invFun`, also written `e.toEquiv.symm`, is the inverse of that same e. The second field says $$e^{-1}(c_{\mathcal{K}(K)}(F)) \subseteq c_{C_A(I)}(e^{-1}(F))$$ for every $F \subseteq S(\mathcal{K}(K))$. Together the two fields give the arbitrary-set closure equality (2).
Exact declaration Continuity definition `def Continuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop := ∀ s : Set α, f '' c.closure s ⊆ d.closure (f '' s)`	For a map $r : Y \to Z$, continuity means $$r(c_Y(E)) \subseteq c_Z(r(E))$$ for every $E \subseteq Y$. Source `α`, `β`, `c`, `d`, `f` denote Y, Z, c_Y, c_Z, r; the double prime denotes the set image $r(E)$. Unlike `FinitelyContinuous`, this definition has no finiteness restriction.

Size scope: the principal equivalence fixes the universe of A and quantifies both representation spaces and compact-model spaces in that same universe. The separately cited transport theorem allows an independently specified Hilbert-space universe; it does not change the quantifiers of this declaration.

Exact surrounding source context (separate excerpts)

HomogeneousCharacterization.lean.

```
variable {A : Type u} [NonUnitalCStarAlgebra A] [hA : Nontrivial A]
attribute [local instance] _root_.CStarAlgebra.spectralOrder
  _root_.CStarAlgebra.spectralOrderedRing
```

HomogeneousCharacterization.lean.

```
include hA in
```

Exact declaration for the source guide

```
theorem forall_irreducible_finrank_eq_iff_component_compact_homeomorph (n : ℕ) :
    (∀ (L : Type u) [NormedAddCommGroup L] [InnerProductSpace ℂ L] [CompleteSpace L],
      ∀ pi : NonUnitalCStarRepresentation A L, pi.IsIrreducible →
        FiniteDimensional ℂ L ∧ Module.finrank ℂ L = n) ↔
    ∀ I : NormedSpace.GeometricStructureSpace ℂ A,
      ClosureSpace.HasFiniteDimensionalCompactHomeomorph.{u, u}
        ((NormedSpace.geometricStructureClosure ℂ A).subspace
          ((NormedSpace.geometricStructureClosure ℂ A).component I)) (fun m => m = n) :=
```