

The separable trace-GNS space is infinite-dimensional

`MathlibAnnex.CStarAlgebra.CAR.not_finiteDimensional_separableCounterexampleHilbertSpace`

Rules out a finite-dimensional Hilbert space for the faithful tracial model.

Statement

The CAR trace-GNS Hilbert space H_τ carrying the specified faithful representation of the fixed algebra A is infinite-dimensional over $\mathbb{C}$.

Assumptions

Let $A \subseteq B(H_{at})$ be the fixed unital C^* -algebra obtained by adjoining the chosen shell-link unitaries to the atomic representation of the CAR algebra C , and then taking the norm-closed unital $*$ -algebra they generate. Let $H_\tau = L^2(C, \tau_C)$ be the GNS Hilbert space of the normalized CAR trace τ_C . Use the established faithful representation $\rho: A \rightarrow B(H_\tau)$. No finite-dimensionality or CH hypothesis is assumed.

Conclusion

H_τ is not finite-dimensional over $\mathbb{C}$. Together with the separate separability and nontriviality results, it is a nonzero separable infinite-dimensional complex Hilbert space.

Proof route

If H_τ were finite-dimensional, $B(H_\tau)$ would be finite-dimensional. The injective linear map underlying ρ would then make A finite-dimensional, contradicting the counterexample theorem for A .

Proof steps

1. Assume finite-dimensionality of H_τ and obtain the corresponding instance for $B(H_\tau)$.
2. Apply the finite-dimensional-subspace argument to the injective representation ρ .
3. Contradict the established infinite-dimensionality of A .

Lean realization notes

The exact conclusion is $\neg \text{FiniteDimensional } \mathbb{C} \text{ Hil}$. A countably infinite orthonormal basis follows by standard Hilbert-space theory from this and separability, not by a new basis-cardinality theorem in the cited declaration. `FiniteDimensional` here is the ordinary finite-dimensional vector-space predicate.

Language: en · [CARD_CONTENT_COMPLETE](#)

Main citations

[MathlibAnnex.CStarAlgebra.CAR.not_finiteDimensional_shellFamilyTarget](#)

[MathlibAnnex.CStarAlgebra.CAR.separableCounterexampleRepresentation_injective](#)

Lean source declaration (exact)

```
theorem not_finiteDimensional_separableCounterexampleHilbertSpace :  
  ¬ FiniteDimensional ℂ SeparableCounterexampleHilbertSpace := by  
  intro hfinite  
  letI : FiniteDimensional ℂ SeparableCounterexampleHilbertSpace := hfinite  
  letI : FiniteDimensional ℂ  
    (SeparableCounterexampleHilbertSpace → L[ℂ] SeparableCounterexampleHilbertSpace) :=  
    ContinuousLinearMap.finiteDimensional  
  exact (not_finiteDimensional_shellFamilyTarget homogeneityShellFamily)  
    (FiniteDimensional.of_injective  
      (LinearMapClass.linearMap_separableCounterexampleRepresentation)  
      separableCounterexampleRepresentation_injective)
```

Read exact [MathlibAnnex v0.4.0 source](#)

Exact Card identity

Stable Card ID: [acd238bacdf236617d1b6fcee52bc6c12e193116f911bbb8f2020a1c9979dc8d](#)

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