The selected GNS representation of a pure-state class

`MathlibAnnex.CStarAlgebra.PureState.selectedRepresentation`

def

A choice of representative pure states, fixed literally at a root, gives one concrete GNS representation per equivalence class.

Statement

Let A be a unital complex C*-algebra, with the positive order used for its states, and fix a pure state ω. A pure state is an extreme point of the normalized positive continuous linear functionals. Let I be the set of pure-state GNS representations modulo unitary equivalence. For each $j \in I$, choose a pure state φ_j in that class, retaining $\varphi_o = \omega$ literally at $o = [\omega]$. Write (H_j, π_j, ξ_j) for its GNS space, representation and canonical cyclic vector. The selected representation is the GNS action $\pi_j : A \to B(H_j)$ associated to this particular φ_j. The choice concerns the representative state; the construction then uses its actual GNS Hilbert space and action.

Definition

Take the chosen state φ_j, regard it as a positive linear functional, and form its GNS completion H_j. Left multiplication induces the bounded action π_j on that completion. The selected vector ξ_j is the GNS cyclic vector. Its orbit $\pi_j(A)\xi_j$ is dense. The vector-functional identity and purity of φ_j, together with this cyclicity and normalization, supply the separate irreducibility theorem.

Assumptions

The class $j \in I$ and the root pure state ω are given. GNS representations here are unital complex-linear star homomorphisms on complete complex Hilbert spaces. No separability or faithfulness of π_j is assumed.

Conclusion

The definition specifies π_j on H_j. The separate selected-vector and pure-GNS results give $\|\xi_j\| = 1$, $\varphi_j(a) = \langle \xi_j, \pi_j(a)\xi_j \rangle$, and irreducibility of π_j. These properties explain why this selected family supplies the summands of the direct-sum representation.

Main citations

- Definition and its exact construction · Exact source
- Root-preserving choice of a pure state in each GNS class · Exact source
- The selected GNS Hilbert space · Exact source
- The selected cyclic vector · Exact source
- Normalization of the selected cyclic vector · Exact source
- The selected state as a vector functional · Exact source
- Density of the selected GNS orbit · Exact source
- Irreducibility supplied by purity and cyclicity · Exact source

Lean source signature (exact)

```
noncomputable def selectedRepresentation (root : PureState A) (j : GNSClass A) :
    MathlibAnnex.Analysis.CStarAlgebra.Representation A (SelectedGNS root j) :=
  (representative root j).positiveFunctional.gnsStarAlgHom
```

> Here `root` is ω, `j` is a GNS equivalence class, `representative root j` is φ_j, and `SelectedGNS root j` is H_j. The RHS `positiveFunctional.gnsStarAlgHom` is π_j. The vector ξ_j is named `selectedVector root j` in its separate cited definition; the normalization, vector-functional identity and irreducibility are separate cited theorems, not extra fields of this definition.

Lean realization notes

The inner product is conjugate linear in its first argument. Irreducibility means that the only closed reducing subspaces are 0 and H_j; this Hilbert space is nonzero because it contains the unit vector ξ_j. The definition does not itself prove those properties, and it does not claim that different pure states always have inequivalent GNS representations.

▼ **Exact Card identity**

Language: en
CARD_CONTENT_COMPLETE
Card UID: lfh:lfh-declaration-card:sha256:ad01be2e337dc8b82ab777026820a387f45a7b982625ae89324c84e796897a26
Card revision: 1 · SHA-256: dc502af1f12012931543b84153b5333bda1d2d5ab24463cec516f75c4482f1ee
Exposition revision: 1 · SHA-256: af9004dce076d51049a2e04d5af836787c064af87a8d49609c96985a418fbc9c
Source commit: 437e6e46228bbb8e91211ebded349d7a30020e73
Header SHA-256: 00fdec4eeaae55b967c9b4eb57d8e7b5d9807fcf7965ba20f902009f90337bcf