The same reduced atomic representation and its exact diagonal range

`MathlibAnnex.Analysis.CStarAlgebra.PrimitiveSpectrum.reducedAtomicRepresentation_spec_of_isCompactlyRepresented`

theorem

Collects the simultaneous properties and exact diagonal range of the same reduced atomic representation.

Statement

Let A be a complex C*-algebra, let H be a complex Hilbert space, and let $\pi : A \to \mathcal{B}(H)$ be a faithful $*$-representation with $\pi(A) \subseteq \mathcal{K}(H)$. Here $\mathcal{K}(L)$ denotes the algebra of compact operators on a complex Hilbert space L. Write $\mathrm{Prim}(A)$ for the primitive ideal spectrum, whose points are the kernels of pure-state GNS representations. For each $P \in \mathrm{Prim}(A)$, fix the pure state φ_P selected by the primitive-point representative. Its GNS representation satisfies $\ker \pi_{\varphi_P} = P$ by the kernel-identification theorem. Write H_P for its GNS Hilbert space and $\rho_P = \pi_{\varphi_P}$. On $\mathcal{H} = \bigoplus^2_P H_P$, the reduced atomic representation $R : A \to \mathcal{B}(\mathcal{H})$ is specified by the coordinate-action theorem:

$$(R(a)\xi)_P = \rho_P(a)\xi_P \qquad (a \in A,\ \xi \in \mathcal{H}).$$

For a bounded family of compact operators $T = (T_P)$ with $T_P \in \mathcal{K}(H_P)$, let $\Delta_b(T)$ act diagonally on $\mathcal{H}$:

$$(\Delta_b(T)\xi)_P = T_P\xi_P, \qquad \sup_P \|T_P\| < \infty.$$

Write Δ_0 for the restriction to the c_0-sum: for every $\varepsilon > 0$, the set $\{P : \|T_P\| \geq \varepsilon\}$ is finite. These bounded diagonal and c0-diagonal action formulas refer to the same Hilbert sum. Then, simultaneously:

- R is injective and compact-valued;
- $\operatorname{ran} R = \operatorname{ran} \Delta_0$, where Δ_0 is the diagonal action of $\bigoplus^{c_0}_P \mathcal{K}(H_P)$;
- $\operatorname{ran} R = \operatorname{ran} \Delta_b \cap \mathcal{K}(\mathcal{H})$, where Δ_b is the diagonal action of the bounded product of compact blocks;
- R is an isometry and $\|R(a)\| = \|a\|$ for every $a \in A$;
- for the coordinate projection p_P and embedding j_P, $p_P R(a) j_P = \rho_P(a)$;
- $R(a)^* = R(a^*)$ for every $a \in A$.

Assumptions

The given π acts on the complete complex Hilbert space H. Only its injectivity and pointwise compactness are assumed. No irreducibility, nonzero algebra, nonempty index, countability, or separability is imposed. The Hilbert sum and its selected fibers are the already defined ones.

Conclusion

All outputs concern this one R, not a replacement action. Its range is block diagonal; the theorem does not assert that it is all of $\mathcal{K}(\mathcal{H})$ or that arbitrary off-diagonal compact operators are present. It also does not claim unitary equivalence between R and the original π.

Main citations

The simultaneous reduced-atomic specification concerns the already defined reduced-atomic map. Its range as a compact c0-diagonal action and range as the compact part of the bounded diagonal action are exact equalities for that same map, not the whole compact-operator algebra.

Lean source signature (exact)

Exact source and declaration. Only the literal signature is displayed here; the linked source retains the proof.

```
theorem reducedAtomicRepresentation_spec_of_isCompactlyRepresented
    (pi : NonUnitalCStarRepresentation A H) (hpi : pi.IsCompactlyRepresented) :
    (reducedAtomicRepresentation (A := A)).IsCompactlyRepresented ∧
    (NonUnitalStarAlgHom.range (reducedAtomicRepresentation (A := A))) =
      (NonUnitalStarAlgHom.range (CompactOperatorC0Sum.diagonalRepresentation
        (fun P : PrimitiveSpectrum A => P.pureStateRepresentative.GNS))) ∧
    (NonUnitalStarAlgHom.range (reducedAtomicRepresentation (A := A))) =
      (NonUnitalStarAlgHom.range (CompactOperatorBoundedProduct.diagonalRepresentation
        (fun P : PrimitiveSpectrum A => P.pureStateRepresentative.GNS))) ⊓
       ContinuousLinearMap.compactOperatorStarSubalgebra ℂ
         (ReducedAtomicHilbertSpace A) ∧
    Isometry (reducedAtomicRepresentation (A := A)) ∧
    (∀ a : A, ‖reducedAtomicRepresentation (A := A) a‖ = ‖a‖) ∧
    (∀ (a : A) (P : PrimitiveSpectrum A),
      (coordinateProjection P).comp
        ((reducedAtomicRepresentation (A := A) a).comp (coordinateEmbedding P)) =
          P.pureStateRepresentative.gnsRepresentation a) ∧
    (∀ a : A, ContinuousLinearMap.adjoint (reducedAtomicRepresentation (A := A) a) =
      reducedAtomicRepresentation (star a))
```

In the source	Mathematical meaning

Algebra, Hilbert space and positivity
`universe u v`
 	
`variable {A : Type u} [NonUnitalCStarAlgebra A]`
`variable {H : Type v} [NormedAddCommGroup H]`
`[InnerProductSpace ℂ H] [CompleteSpace H]`
	The complex C*-algebra A and complex Hilbert space H are independent inputs. Completeness is part of the Hilbert structure. The local spectral-order instances displayed separately provide the positivity used for pure states.

Given faithful compact action
`(pi : NonUnitalCStarRepresentation A H) (hpi :`
`pi.IsCompactlyRepresented) :`
	The input is the given $\pi : A \to \mathcal{B}(H)$, faithful with $\pi(A) \subseteq \mathcal{K}(H)$. This hypothesis places no irreducibility or nonzero requirement on A.

In the source	Mathematical meaning
Meaning of compact representability `def NonUnitalCStarRepresentation.IsCompactlyRepresented (pi : NonUnitalCStarRepresentation A H) : Prop := Function.Injective pi ∧ ∀ a : A, IsCompactOperator (pi a)`	Injectivity is faithfulness, and the universal compactness condition says that every $\pi(a)$ is compact. The same definition is applied to R in the first conclusion.
Fixed pure-state representatives `def pureStateRepresentative (P : PrimitiveSpectrum A) : NonUnitalPureState A := P.property.choose`	For each $P \in \mathrm{Prim}(A)$ this selects the fixed pure state φ_P. The separately cited kernel theorem identifies $\ker \rho_P = P$, where $\rho_P = \pi_{\varphi_P}$ on H_P. These are the same representatives in every conclusion.
Hilbert sum `abbrev ReducedAtomicHilbertSpace (A : Type u) [NonUnitalCStarAlgebra A] [PartialOrder A] [StarOrderedRing A] := HilbertSum (fun P : PrimitiveSpectrum A => P.pureStateRepresentative.GNS)`	The representation space is $$\mathcal{H} = \bigoplus_{P \in \mathrm{Prim}(A)}^{2} H_P.$$ This is the Hilbert direct sum of the selected GNS spaces, not the algebra of bounded operator families.
Reduced atomic action `theorem reducedAtomicRepresentation_apply (a : A) (x : ReducedAtomicHilbertSpace A) (P : PrimitiveSpectrum A) : reducedAtomicRepresentation a x P = P.pureStateRepresentative.gnsRepresentation a (x P) := by`	The coordinate formula specifies the same $R : A \to \mathcal{B}(\mathcal{H})$: $$(R(a)\xi)_P = \rho_P(a)\xi_P.$$ It holds for every $a \in A$, $\xi \in \mathcal{H}$ and $P \in \mathrm{Prim}(A)$.
Faithfulness and compact values of R `(reducedAtomicRepresentation (A := A)).IsCompactlyRepresented ∧`	The first conclusion is that this R is injective and that $R(a)$ is compact for every $a \in A$.
Range as c0 diagonal operators `(NonUnitalStarAlgHom.range (reducedAtomicRepresentation (A := A))) = (NonUnitalStarAlgHom.range (CompactOperatorC0Sum.diagonalRepresentation (fun P : PrimitiveSpectrum A => P.pureStateRepresentative.GNS))) ∧`	The family argument assigns the same H_P to each primitive ideal. The equality is $$\mathrm{ran}\, R = \mathrm{ran}\, \Delta_0,$$ where Δ_0 acts on the c_0-family of compact blocks defined in the Statement, on the same $\mathcal{H}$.
Range as compact bounded-diagonal operators `(NonUnitalStarAlgHom.range (reducedAtomicRepresentation (A := A))) = (NonUnitalStarAlgHom.range (CompactOperatorBoundedProduct.diagonalRepresentation (fun P : PrimitiveSpectrum A => P.pureStateRepresentative.GNS))) ⊓ ContinuousLinearMap.compactOperatorStarSubalgebra ℂ (ReducedAtomicHilbertSpace A) ∧`	The meet ⊓ is intersection of subalgebras of $\mathcal{B}(\mathcal{H})$. Thus $$\mathrm{ran}\, R = \mathrm{ran}\, \Delta_b \cap \mathcal{K}(\mathcal{H}).$$ The bounded family and the compact-operator algebra use the same selected Hilbert sum; no off-diagonal operators are added.

In the source	Mathematical meaning
Isometry and norm equality `Isometry (reducedAtomicRepresentation (A := A)) ∧ (∀ a : A, ‖reducedAtomicRepresentation (A := A) a‖ = ‖a‖) ∧`	The same R preserves distances and satisfies $$\|R(a)\| = \|a\| \qquad (a \in A).$$ Both assertions occur explicitly as conjuncts.
Coordinate compression `(∀ (a : A) (P : PrimitiveSpectrum A), (coordinateProjection P).comp ((reducedAtomicRepresentation (A := A) a).comp (coordinateEmbedding P)) = P.pureStateRepresentative.gnsRepresentation a) ∧`	The projection $p_P : \mathcal{H} \to H_P$ and inclusion $j_P : H_P \to \mathcal{H}$ give $$p_P R(a) j_P = \rho_P(a) \qquad (a \in A,\ P \in \mathrm{Prim}(A)).$$ The order of the two `comp` operations is exactly this composition.
Adjoint identity `(∀ a : A, ContinuousLinearMap.adjoint (reducedAtomicRepresentation (A := A) a) = reducedAtomicRepresentation (star a)) := by`	The last conjunct is $$R(a)^* = R(a^*) \qquad (a \in A).$$ The left side is the Hilbert adjoint on $\mathcal{H}$ and the right side uses the involution of A. It concerns the same R as all preceding conjuncts.

Exact surrounding source context (separate excerpts)

Exact surrounding source:

```
attribute [local instance] _root_.CStarAlgebra.spectralOrder
  _root_.CStarAlgebra.spectralOrderedRing
```

Exact surrounding source:

```
universe u v

variable {A : Type u} [NonUnitalCStarAlgebra A]
variable {H : Type v} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
```

Related exact source for the reading guide

Meaning of compact representability · Fixed source

```
def NonUnitalCStarRepresentation.IsCompactlyRepresented
    (pi : NonUnitalCStarRepresentation A H) : Prop :=
  Function.Injective pi ∧ ∀ a : A, IsCompactOperator (pi a)
```

Hilbert sum · Fixed source

```
abbrev ReducedAtomicHilbertSpace (A : Type u) [NonUnitalCStarAlgebra A]
    [PartialOrder A] [StarOrderedRing A] :=
  HilbertSum (fun P : PrimitiveSpectrum A => P.pureStateRepresentative.GNS)
```

Reduced atomic action · Fixed source

```
theorem reducedAtomicRepresentation_apply
    (a : A) (x : ReducedAtomicHilbertSpace A) (P : PrimitiveSpectrum A) :
    reducedAtomicRepresentation a x P =
      P.pureStateRepresentative.gnsRepresentation a (x P) := by
```

Adjoint identity · Fixed source

```
    (∀ a : A, ContinuousLinearMap.adjoint (reducedAtomicRepresentation (A := A) a) =
      reducedAtomicRepresentation (star a)) := by
```

Fixed pure-state representatives · Fixed source

```
def pureStateRepresentative (P : PrimitiveSpectrum A) : NonUnitalPureState A :=
  P.property.choose
```