The closed operator algebra generated by a representation and extra operators

`MathlibAnnex.CStarAlgebra.AtomicConstruction.concreteTarget`

def

Places a represented algebra and a chosen family of bounded operators in one concrete closed algebra.

Statement

Let A be a unital complex algebra with an additive, multiplicatively reversing involution, and let H be a complex Hilbert space. Write $B(H)$ for its bounded complex-linear operators. Fix a unital complex-linear $*$-homomorphism $\pi : A \to B(H)$, that is, a representation satisfying $\pi(a^*) = \pi(a)^*$. For any family $(U_j)_{j \in J}$ in $B(H)$, form the norm-closed unital $*$-subalgebra generated by $\pi(A)$ and the U_j. This is the concrete target T.

Definition

Put $S = \pi(A) \cup \{U_j : j \in J\}$. Let $\operatorname{alg}^*_{\mathbb{C}}(S)$ denote the smallest unital complex subalgebra of $B(H)$ containing S and closed under adjoints. Define $T = \overline{\operatorname{alg}^*_{\mathbb{C}}(S)}$, where the bar denotes operator-norm closure. Thus T consists of operator-norm limits of finite complex linear combinations of finite products of members of S and their adjoints, including the empty product 1. The adjoint in $B(H)$ and the algebra operations restrict to T.

Assumptions

The algebra A is unital and associative over $\mathbb{C}$, with an additive involution satisfying $(ab)^* = b^*a^*$ and $(a^*)^* = a$. The map π preserves multiplication, the unit, complex scalars and this involution. The space H is complete for its complex inner-product norm. The index set J is arbitrary. No norm on A, faithfulness of π, or conjugate-linearity of the involution on A is assumed.

Conclusion

The construction provides a closed unital star subalgebra $T \subseteq B(H)$ containing the displayed operators. Write $\iota : A \to T$ for $\iota(a) = \pi(a)$, $g_j \in T$ for the element represented by U_j, and $\rho : T \to B(H)$ for inclusion. Then $\rho \circ \iota = \pi$ and $\rho(g_j) = U_j$. The inclusion ρ is injective; if π is injective, so is ι. These maps and their injectivity statements are the contextual facts cited below, not additional clauses of the defining declaration.

Main citations

- Exact declaration and proof · Exact source
- MathlibAnnex.Analysis.CStarAlgebra.generated · Exact source
- MathlibAnnex.CStarAlgebra.AtomicConstruction.sourceHom · Exact source
- MathlibAnnex.CStarAlgebra.AtomicConstruction.sourceHom_coe · Exact source
- MathlibAnnex.CStarAlgebra.AtomicConstruction.generator · Exact source
- MathlibAnnex.CStarAlgebra.AtomicConstruction.generator_coe · Exact source
- MathlibAnnex.CStarAlgebra.AtomicConstruction.ambientInclusion · Exact source
- MathlibAnnex.CStarAlgebra.AtomicConstruction.sourceHom_injective · Exact source
- MathlibAnnex.CStarAlgebra.AtomicConstruction.ambientInclusion_injective · Exact source
- MathlibAnnex.CStarAlgebra.AtomicConstruction.concreteTarget.instIsClosed · Exact source

Lean source signature (exact)

```
noncomputable def concreteTarget (pi : A →⋆ₐ[ℂ] (H →L[ℂ] H))
    (U : J → H →L[ℂ] H) : StarSubalgebra ℂ (H →L[ℂ] H) :=
  generated (Set.range pi ∪ Set.range U)
```

> In the code, `pi` is π, `U` is the family (U_j), and `concreteTarget pi U` is T. The notation `H →L[ℂ] H` means a bounded complex-linear operator on H; `→⋆ₐ[ℂ]` denotes a unital complex-linear $*$-homomorphism.

Lean realization notes

The closure is taken inside the given operator algebra. The construction does not assert simplicity or irreducibility, and it introduces no universal completion of A. In particular, an arbitrary extra operator U_j need not be unitary.

▼ **Exact Card identity**

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