

Uniqueness of the tracial state

MathlibAnnex.CStarAlgebra.CAR.existsUnique_tracial_state

Proves uniqueness among normalized positive traces on each shell-family algebra.

Statement

For every representative shell family F , the associated C^* -algebra A_F has exactly one tracial state.

Assumptions

Let C be the completed CAR algebra, ϕ_0 its distinguished pure state, ϕ_i one selected pure state in each GNS-equivalence class, and e_n the fixed root shell projections. Fix a representative shell family F : for each i , an automorphism α_i and elements $w_{i,n}$ of C satisfying $\phi_i \circ \alpha_i = \phi_0$, $w_{i,n}^* w_{i,n} = \alpha_i(e_n)$, and $w_{i,n} w_{i,n}^* = e_n$. At the root class, α_0 is the identity and $w_{0,n} = e_n$. Let $A_F \subseteq B(H_{\text{at}})$ be the norm-closed unital $*$ -algebra generated by the atomic image of C and the shell-link unitaries constructed from F ; write $j_F: C \rightarrow A_F$ for the source map. States are continuous positive complex-linear functionals normalized by $\phi(1) = 1$.

Conclusion

There exists exactly one state $\phi: A_F \rightarrow \mathbb{C}$ satisfying $\phi(ab) = \phi(ba)$ for all $a, b \in A_F$. It is the constructed extension of the normalized CAR trace.

Proof route

The constructed extension is tracial. Any tracial state on A_F restricts to the unique normalized CAR trace, so uniqueness of the state extension identifies it with the constructed extension.

Proof steps

1. Use the separate theorem proving traciality of the constructed extension.
2. Restrict an arbitrary tracial state along j_F and apply uniqueness of the CAR trace.
3. Use the uniqueness theorem for extensions of the CAR trace to identify that state on all of A_F .

Lean realization notes

The theorem states unique existence of a state satisfying the trace identity, not uniqueness of all states. Its displayed conclusion does not itself assert faithfulness. The latter is proved separately and included for the fixed A in the counterexample theorem with its faithful separable tracial model.

Language: en · CARD_CONTENT_COMPLETE

Main citations

[MathlibAnnex.CStarAlgebra.CAR.eq_traceExtension_of_mem_stateSpace_of_mul_comm](#)

Lean source declaration (exact)

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/-- The existing target has exactly one tracial state. Faithfulness is proved
separately above; uniqueness ranges over all states, not just selected extensions. -/
theorem existsUnique_tracial_state (family : RepresentativeShellFamily) :
  ∃! φ : ShellFamilyTarget family → L[ℂ] ℂ,
  φ ∈ MathlibAnnex.Analysis.CStarAlgebra.stateSpace (ShellFamilyTarget family) ∧
  ∀ a b, φ (a * b) = φ (b * a) := by
  refine {traceExtension family,
    {traceExtension_mem_stateSpace family, traceExtension_mul_comm family}, ?_}
  intro φ hφ
  exact eq_traceExtension_of_mem_stateSpace_of_mul_comm family φ hφ.1 hφ.2
```

[Read exact MathlibAnnex v0.4.0 source](#)

Exact Card identity

Stable Card ID: b236c960b8ba284e9d9a4316ce2b081f991bd025a1efb7fb735342b686e38cee

Card SHA-256: b44fc1eb359af758468f55442e998fc4f91a92be8dcf566311f37e8b000bc06f

Approved exposition SHA-256: 08b702ddb0ef4d093782c0e54152773cdfec1ef8a21608b8d165c1b9160478b

AI-assisted material. Selected, edited, and reviewed by Ryotaro Tanaka. The recorded review is source–exposition correspondence, not an independent proof audit.