The fixed shell target has a unique tracial state

`MathlibAnnex.CStarAlgebra.CAR.existsUnique_tracial_state`

theorem

Every tracial state is forced back to the CAR trace and hence to its unique extension.

Statement

Let A be the completed CAR algebra and τ its normalized trace. Fix a representative shell family and its chosen unitary links L_j in the selected atomic representation π. Write $T = C^*(\pi(A), \{L_j\})$ for the resulting concrete target and $\iota : A \to T$, $\iota(a) = \pi(a)$, for its faithful unital source map. This fixes one target and one source map throughout. There exists exactly one state $\phi : T \to \mathbb{C}$ such that $\phi(ab) = \phi(ba)$ for all $a, b \in T$.

Assumptions

The representative shell family is fixed. A competing functional is required to be a state and tracial on T; equality with τ on the source is derived, not assumed.

Conclusion

The unique tracial state is the chosen extension `traceExtension family`. This uniqueness quantifies all tracial states on T, whereas extension uniqueness quantifies all states with the prescribed source restriction.

Proof route

Produce the tracial extension, restrict any other tracial state to the source, and invoke the two uniqueness results in that order.

Proof steps

1. The extension theorem gives a state ϕ with $\phi \circ \iota = \tau$, and `traceExtension_mul_comm` proves it tracial on T. This supplies the existence part.

2. For any tracial state ψ on T, the functional $f = \psi \circ \iota$ is continuous and complex linear. Since ι preserves 1 and products,

   $$f(1) = 1, \qquad f(ab) = \psi(\iota(a)\iota(b)) = \psi(\iota(b)\iota(a)) = f(ba).$$

   CAR trace uniqueness, `eq_trace_of_apply_one_of_mul_comm`, therefore gives $f = \tau$. That source theorem determines values on matrix units and extends from finite stages by density; no target-separability hypothesis is used.

3. Now ψ is a state extension of τ. The extension-equality theorem applies and gives $\psi = \phi$. In the source, these two steps are `apply_shellFamilySourceHom_eq_trace_of_apply_one_of_mul_comm` followed by `eq_traceExtension_of_mem_stateSpace_of_mul_comm`.

Main citations

- The stated existence or structural result · Exact source

- The fixed concrete target and source map · Exact source
- Faithfulness of the source embedding · Exact source
- The chosen extension is tracial · Exact source
- Uniqueness with prescribed source trace · Exact source
- Uniqueness of the normalized continuous CAR trace · Exact source
- Restriction of any target trace is the CAR trace · Exact source
- Identification of an arbitrary tracial state · Exact source
- Separate state-faithfulness assertion · Exact source

Lean source signature (exact)

```
theorem existsUnique_tracial_state (family : RepresentativeShellFamily) :
    ∃! ϕ : ShellFamilyTarget family →L[ℂ] ℂ,
      ϕ ∈ MathlibAnnex.Analysis.CStarAlgebra.stateSpace (ShellFamilyTarget family) ∧
        ∀ a b, ϕ (a * b) = ϕ (b * a)
```

> `∃! ϕ` ranges over continuous functionals on `ShellFamilyTarget family`; the predicate includes state-space membership and the identity for all `a b`. In contrast to the extension-uniqueness signature, it does not prescribe source values. The proof derives those values using CAR trace uniqueness before applying extension uniqueness.

Lean realization notes

Faithfulness of a representation and faithfulness of this state are different assertions. This theorem asserts existence and uniqueness of a tracial state. The separate result `traceExtension_star_mul_self_eq_zero_iff` concerns state faithfulness and is not an additional hypothesis here.

▼ **Exact Card identity**

Language: en
CARD_CONTENT_COMPLETE
Card UID: lfh:lfh-declaration-card:sha256:b236c960b8ba284e9d9a4316ce2b081f991bd025a1efb7fb735342b686e38cee
Card revision: 2 · SHA-256: b44fc1eb359af758468f55442e998fc4f91a92be8dcf566311f37e8b000bc06f
Exposition revision: 3 · SHA-256: 0bd44fead9f058ef6ab97fcb1774f55d403cba3f128d6d1f602d5374423d3282
Source commit: 437e6e46228bbb8e91211ebded349d7a30020e73
Header SHA-256: d9616be03bf1f37107e291058c9b4567c3482f114198ecb6d4b99c687513b78c