The irreducible spectrum and the geometric spectrum

`MathlibAnnex.Analysis.CStarAlgebra.irreducibleSpectrumHomeomorph`

def

Identifies irreducible representation classes with components of the geometric structure space.

Statement

Let $A \neq \{0\}$ be a complex C*-algebra, possibly without an identity. There is a specified homeomorphism

$$\Phi_A : \widehat{A} \longrightarrow \mathrm{GS}(A)$$

from its independent irreducible representation spectrum to its geometric spectrum. For a pure state φ and its GNS representation π_φ, its point map is

$$\Phi_A([\pi_\varphi]) = C(\ker \varphi). \tag{1}$$

Here $S(A)$ consists of kernels of extreme points of the closed unit ball of the Banach dual of A, with convexity over $\mathbb{R}$. The extreme-functional kernel characterization identifies this family with the original geometric structure space, with closure $c_A(\mathcal{U}) = \{I : \bigcap_{J \in \mathcal{U}} J \subseteq I\}$. The component is taken for the finite-chain relation generated by equal points and nonclosed pairs.

Definition

The source $\widehat{A}$ is the irreducible spectrum: pure norm-one positive bounded complex-linear functionals on A, modulo unitary equivalence of their GNS representations restricted to A. Equality of representation kernels is not the equivalence relation. Its topology is induced by the kernel map $\kappa([\pi]) = \ker \pi$ into the primitive spectrum with hull-kernel topology.

The target $\mathrm{GS}(A)$ consists of components of $S(A)$. For a component C, put $J_C = \bigcap_{I \in C} I$. Its closure is

$$\overline{\mathcal{D}} = \left\{ C : \bigcap_{D \in \mathcal{D}} J_D \subseteq J_C \right\},$$

with empty intersection A. The geometric-spectrum topology has this closure.

The spectral bijection sends $[\pi_\varphi]$ to $C(\ker \varphi)$. Its inverse assigns the representation class of a pure state whose functional kernel lies in the given component, independently of the chosen pure state. The closure-preservation theorem applies to every subset. The resulting closure-space homeomorphism, for the two specified ordinary topologies, is Φ_A.

Assumptions

The algebra is nonzero and has the compatible C*-order required by the pure-state and primitive-kernel constructions. The ordinary topology on $\widehat{A}$ is the kernel-induced topology just described; no topology from a quotient of the pure-state space is substituted. No injectivity of κ is imposed.

Conclusion

The output is the specified ordinary homeomorphism with action (1), for the kernel-induced topology on $\widehat{A}$ and the stated topology on $\mathrm{GS}(A)$. Its inverse is the inverse of this same bijection. No injectivity of the kernel map into the primitive spectrum is asserted.

Main citations

- Exact full definition: `irreducibleSpectrumHomeomorph`.
- Independent spectrum and topology, with its point construction in Spectrum/Basic.lean: the source of the displayed homeomorphism.
- SpectralComparison.lean, `toGeometricSpectrum_ofRepresentation`: the formula for any irreducible action and any nonzero coefficient pair.
- Ryotaro Tanaka, arXiv:2504.05551v1, Theorem 5.11 and Corollary 5.12: spectral reconstruction and the ordinary topological conclusion.
- image_irreducibleSpectrumClosure: full closure preservation.
- irreducibleSpectrumClosureHomeomorph: the bundled map and inverse.

Lean source signature (exact)

The full definition below is copied from the pinned source.

```
def irreducibleSpectrumHomeomorph :
    @_root_.Homeomorph (IrreducibleSpectrum A) (NormedSpace.GeometricSpectrum ℂ A)
      inferInstance (geometricSpectrumTopology A) :=
  (irreducibleSpectrumClosureHomeomorph A).toTopologicalHomeomorph
    (isTopologizable_irreducibleSpectrumClosure A) (isTopologizable_geometricSpectrumClosure A)
```

In the source	Mathematical meaning
Exact declaration `A` and the surrounding instances	The nonzero nonunital complex C*-algebra with compatible order.
Exact declaration `IrreducibleSpectrum A`	The pure-GNS representation classes, with the topology induced by their representation kernels; not the quotient topology of pure states.

In the source	Mathematical meaning
Exact declaration `NormedSpace.GeometricSpectrum ℂ A`	Components of the geometric structure space, with the ordinary topology having the stated component-intersection closure.
Exact declaration `irreducibleSpectrumClosureHomeomorph A`	The specific bijection $[\pi_\varphi] \mapsto C(\ker \varphi)$ with its inverse and full closure-continuity data.
Exact declaration `.toTopologicalHomeomorph`	Uses the two named topologizability guarantees, without replacing either closure by a new one.

Exact surrounding source context (separate excerpts)

Source lines 28–28:

```
variable (A : Type u) [NonUnitalCStarAlgebra A] [PartialOrder A] [StarOrderedRing A]
```

Source lines 29–29:

```
variable [Nontrivial A]
```