Unitary equivalence without an initial unitality assumption

`MathlibAnnex.CStarAlgebra.CAR.shellFamilyInclusion_unitaryEquivalent_nonUnital`

theorem

Shows why allowing nonunital input representations does not enlarge the irreducible representation class of the shell-generated algebra.

Statement

Let $T \subseteq B(\mathcal{H})$ be the fixed shell-generated algebra below. If $\rho : T \to B(K)$ is a nonzero irreducible complex-linear star representation, then a surjective complex-linear isometry $U : \mathcal{H} \to K$ satisfies $U(\rho_0(t)x) = \rho(t)(Ux)$ for every $t \in T$ and $x \in \mathcal{H}$. Preservation of the unit is a consequence, not an additional hypothesis on ρ.

Assumptions

Let A be the completed CAR algebra with matrix stages $M_{2^n}(\mathbb{C})$ embedded by $a \mapsto a \otimes I_2$. Write f_n for the image of the first diagonal matrix unit, $p_n = f_n - f_{n+1}$, and ω for the root state characterized by $\omega(\iota_n(a)) = a_{00}$ on each stage, where ι_n is its canonical embedding.

Index chosen pure-state Gelfand-Naimark-Segal (GNS) representations by their unitary-equivalence classes $j \in I$, with root index o and representative states φ_j.

A fixed shell family consists of complex-linear unital star automorphisms α_j and elements $w_{j,n} \in A$ satisfying $\varphi_j(\alpha_j(a)) = \omega(a)$, $w_{j,n}^* w_{j,n} = \alpha_j(p_n)$ and $w_{j,n} w_{j,n}^* = p_n$.

At the root, α_o is the identity and $w_{o,n} = p_n$.

Let $\mathcal{H} = \bigoplus_{j \in I} H_j$ be the Hilbert direct sum of these GNS spaces, $\pi = \bigoplus_j \pi_j$ their representation of A, and $b_j = e_j \xi_j \in \mathcal{H}$ its embedded unit cyclic vector, where $\xi_j \in H_j$ is the selected unit cyclic vector and $e_j : H_j \to \mathcal{H}$ is coordinate inclusion.

Choose once a family of unitary operators $L_j \in B(\mathcal{H})$ from the construction of the shell unitaries: $L_j b_j = b_o$, $L_j \pi(\alpha_j(p_n)) = \pi(w_{j,n})$, and $L_o = I_{\mathcal{H}}$. Here $B(\mathcal{H})$ denotes the bounded complex-linear operators on $\mathcal{H}$.

Put $T = C^*(\pi(A), \{L_j : j \in I\}) \subseteq B(\mathcal{H})$, the norm-closed unital star algebra generated by these operators. Let $\iota : A \to T$ be $\iota(a) = \pi(a)$ with its codomain restricted to T, and let $\rho_0 : T \to B(\mathcal{H})$ be inclusion.

The space K is any complete complex Hilbert space in an independent universe. Irreducibility here explicitly includes $\rho \neq 0$ and the assertion that its closed reducing subspaces are only 0 and K. Nontriviality of K alone is not a replacement for $\rho \neq 0$.

Conclusion

The intertwining equation uses the original map ρ. All its represented operators are retained when one regards it as a unital representation. Thus the displayed inclusion represents every such irreducible class up to unitary equivalence.

Proof route

Put $P = \rho(1)$. It is an idempotent, and its range is closed and reducing for ρ. Nonzeroness makes this range nonzero, so irreducibility makes it all of K; an idempotent is the identity on its range, giving $P = I_K$. The unital capture theorem applies to the same operators, since the unit law has just been established. Its unitary intertwiner therefore satisfies the equation for the original map.

Proof steps

1. For every t, $\rho(1)\rho(t) = \rho(t) = \rho(t)\rho(1)$. Star preservation also controls adjoints, so the range of $\rho(1)$ is reducing.

2. If $\rho(1) = 0$, then every $\rho(t)$ is zero. Irreducibility therefore forces the closed range of this nonzero idempotent to be K, whence $\rho(1) = I_K$.

3. Use the theorem identifying every unital irreducible representation with the fixed inclusion. Rebundling ρ as unital changes no operator, so its resulting unitary equivalence is exactly the claimed one.

Main citations

- Exact declaration and proof · Exact source
- MathlibAnnex.CStarAlgebra.CAR.RepresentativeShellFamily · Exact source
- MathlibAnnex.CStarAlgebra.CAR.exists_completedAtomicShellModel · Exact source
- MathlibAnnex.CStarAlgebra.CAR.isUniqueIrreducibleModel_shellFamilyInclusion · Exact source
- MathlibAnnex.CStarAlgebra.CAR.isUniqueIrreducibleModelAmongNonUnital_shellFamilyInclusion · Exact source
- MathlibAnnex.Analysis.CStarAlgebra.NonUnitalRepresentation.map_one_eq_one_of_isIrreducible · Exact source
- The root state bundled with its purity proof · Exact source

Lean source signature (exact)

```
theorem shellFamilyInclusion_unitaryEquivalent_nonUnital
    (family : RepresentativeShellFamily)
    {K : Type v} [NormedAddCommGroup K] [InnerProductSpace ℂ K]
    [CompleteSpace K]
    (rho : NonUnitalRepresentation
      (A := ShellFamilyTarget family) (H := K))
    (hrho : rho.IsIrreducible) :
    ∃ U :
        MathlibAnnex.CStarAlgebra.PureState.SelectedAtomicHilbert completedRootPureState ≃ₗᵢ[ℂ] K,
      ∀ (a : ShellFamilyTarget family)
        (x : MathlibAnnex.CStarAlgebra.PureState.SelectedAtomicHilbert completedRootPureState),
        U (shellFamilyInclusion family a x) = rho a (U x)
```

> Here `completedRootPureState` is the root state ω, bundled with its purity proof, and `SelectedAtomicHilbert completedRootPureState` is $\mathcal{H}$. `ShellFamilyTarget family` is T and `shellFamilyInclusion family` is ρ_0. `NonUnitalRepresentation` means a complex-linear star representation without a required unit law. Its `IsIrreducible` predicate includes $\rho \neq 0$ and the closed reducing-subspace condition. The existential quantifier supplies the unitary U in the Statement; the proof first derives $\rho(1) = I_K$.

Lean realization notes

The result imposes no separability restriction and does not claim that all representations are irreducible. The shell family is fixed before K is chosen. The zero representation is excluded by the exact irreducibility predicate.

▼ **Exact Card identity**

Language: en

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