# The target represented on the CAR trace GNS space

`MathlibAnnex.CStarAlgebra.CAR.traceModelRepresentation`

def

A fixed pointed unitary transports the target action onto the Hilbert space constructed from CAR alone.

## Statement

Let $A$ be the completed CAR algebra and $\tau$ its normalized trace. Fix a representative shell family and its chosen unitary links $L_j$ in the selected atomic representation $\pi$. Write $T = C^*(\pi(A), \{L_j\})$ for the resulting concrete target and $\iota : A \to T$, $\iota(a) = \pi(a)$, for its faithful unital source map. This fixes one target and one source map throughout. Let $(H_\tau, \pi_\tau, \xi_\tau)$ be the GNS representation of the CAR trace, and let $(H_\phi, \rho_\phi, \xi_\phi)$ be the target-state GNS representation for the chosen extension $\phi$. Fix the pointed unitary $E : H_\tau \to H_\phi$ described below. Define the target representation on $H_\tau$ by

$$\widetilde{\rho}(t) = E^{-1}\rho_\phi(t)E \qquad (t \in T).$$

## Definition

Set $H_\tau = L^2(A, \tau)$, $\xi_\tau = [1]$, and $\pi_\tau(a)[b] = [ab]$. The GNS identities give $\|\xi_\tau\| = 1$ and $\tau(a) = \langle \xi_\tau, \pi_\tau(a)\xi_\tau \rangle$. Thus $H_\tau \neq 0$, and its dense continuous CAR orbit makes it separable. The target GNS has the same source vector functional and a dense source orbit. Pointed cyclic transport therefore gives, and `traceGNSUnitary` chooses once,

$$E\xi_\tau = \xi_\phi, \qquad E\pi_\tau(a) = \rho_\phi(\iota(a))E.$$

Conjugate the whole target action by its inverse. Then $E^{-1}\rho_\phi(\iota(a))E = \pi_\tau(a)$ and

$$\langle \xi_\tau, \widetilde{\rho}(t)\xi_\tau \rangle = \langle E\xi_\tau, \rho_\phi(t)E\xi_\tau \rangle = \phi(t).$$

The dense $\pi_\tau(A)\xi_\tau$ orbit is contained in the target orbit, proving target cyclicity. The defining RHS below uses the inverse of this same chosen $E$.

## Assumptions

The representative shell family and chosen state extension are fixed. Both GNS representations restrict to the same source trace, and both source orbits are dense; source cyclicity in $H_\phi$ is a proved result.

## Conclusion

The new representation has $\widetilde{\rho}(\iota(a)) = \pi_\tau(a)$ and realizes $\phi$ at $\xi_\tau$. Its target orbit is dense. The space $H_\tau$ is nonzero and separable and is defined from CAR alone, independently of the family; the target action still depends on the fixed family.

## Main citations

- Definition and its exact construction · Exact source
- The fixed concrete target and source map · Exact source
- Faithfulness of the source embedding · Exact source
- CAR-only trace Hilbert space · Exact source
- CAR trace cyclic vector · Exact source
- CAR trace GNS action · Exact source
- Unit norm of the CAR vector · Exact source
- Nonzero CAR vector · Exact source
- Nonzero CAR trace Hilbert space · Exact source
- CAR vector-state identity · Exact source
- Dense CAR trace orbit · Exact source
- Separable CAR trace Hilbert space · Exact source
- Existence of the pointed source unitary · Exact source
- The unitary chosen once · Exact source
- Source cyclicity of the target-state GNS · Exact source
- Restriction of the transported target action · Exact source
- Transported vector-state identity · Exact source
- Cyclicity of the transported target action · Exact source

## Lean source signature (exact)

```
noncomputable def traceModelRepresentation (family : RepresentativeShellFamily) :
    Representation (ShellFamilyTarget family) TraceHilbertSpace :=
  (traceGNSUnitary family).symm.conjStarAlgEquiv.toStarAlgHom.comp
    (tracialRepresentation family)
```

> `TraceHilbertSpace`, `traceVector`, and `traceRepresentation` are $H_\tau, \xi_\tau, \pi_\tau$. `TracialHilbertSpace family` is the different space $H_\phi$. `traceGNSUnitary family` maps from $H_\tau$ to $H_\phi$; its `.symm.conjStarAlgEquiv` therefore sends an operator $R$ on $H_\phi$ to $E^{-1}RE$ on $H_\tau$. The complete RHS includes composition with `tracialRepresentation family`.

## Lean realization notes

The direction $E : H_\tau \to H_\phi$ determines the conjugation $E^{-1}\rho_\phi E$. This is one chosen pointed unitary, used throughout the definition and its consequences. Injectivity of this representation is stated separately. The target $T$ itself has not been replaced.


▼ **Exact Card identity**

Language: en
CARD_CONTENT_COMPLETE
Card UID: lfh:lfh-declaration-card:sha256:bd015ee41f91bc2a37747d438f714880780826893cb63ee7e84310a987b0611e
Card revision: 1 · SHA-256: 8a9fbbd1a0305f221e617a417ec288d2563b88c6bb308562369f956067b88ccb
Exposition revision: 1 · SHA-256: 7def775971ef9e1b7577a09e0923b4e8abdf7544c283c1e409fb707725196fa3
Source commit: 437e6e46228bbb8e91211ebded349d7a30020e73
Header SHA-256: ab07f8caf89b3e2a6decfd742272c415341eaf45550bfa8fef3ad47077cae2f3