Every geometric-spectrum component contains a pure-state kernel

`MathlibAnnex.Analysis.CStarAlgebra.exists_pureState_kernel_mem_geometricSpectrum`

theorem

Places the kernel of a pure state in every geometric-spectrum component.

Statement

Let $A \neq \{0\}$ be a complex C*-algebra, possibly without an identity. Its geometric structure space $S(A)$ consists of kernels $\ker \rho \subseteq A$ of bounded complex-linear functionals $\rho : A \to \mathbb{C}$ that are extreme points, for real convexity, of the entire closed dual unit ball B_{A^*}, as stated by the extreme-kernel identification, with closure

$$c_A(\mathcal{T}) = \left\{ I \in S(A) : \bigcap_{J \in \mathcal{T}} J \subseteq I \right\}.$$

Two points are linked if they are equal or their pair is not closed. Their finite-chain equivalence classes are the components. A point C of the geometric spectrum $\mathrm{GS}(A)$ is one such component subset of $S(A)$.

For every $C \in \mathrm{GS}(A)$ there is a pure state φ of A such that

$$\ker \varphi \in C. \qquad (1)$$

Here $\ker \varphi$ is a point of $S(A)$: the pure-state extremality theorem guarantees that φ is extreme in the entire closed dual unit ball.

Assumptions

The source fixes a nonzero complex C*-algebra and its compatible order. The component C is arbitrary; it need not be presented initially by a functional or representation. A pure state is a positive norm-one bounded complex-linear functional $A \to \mathbb{C}$, extreme among positive norm-one functionals for real convexity. The geometric spectrum consists of component **sets**, so (1) places a functional kernel, rather than a functional, in C.

Conclusion

The theorem produces one state together with its purity and the membership of that same state's kernel in the specified component. It does not identify $\ker \varphi$ with the intersection of all points of C; that intersection is determined in the component-kernel identity.

Main citations

- Exact component representative theorem: `exists_pureState_kernel_mem_geometricSpectrum`.
- Ryotaro Tanaka, arXiv:2504.05551v1, Corollary 5.9: the component representative assertion, read with the type-correct pure-state **kernel**.

Lean source signature (exact)

The declaration prefix below is a contiguous source excerpt, ending before its proof assignment.

```
theorem exists_pureState_kernel_mem_geometricSpectrum (C : NormedSpace.GeometricSpectrum ℂ A) :
    ∃ (phi : StrongDual ℂ A) (hpure : IsPureNonUnitalState A phi),
      extremeKernel ⟨phi, mem_extremePoints_dual_closedBall_of_isPureNonUnitalState phi hpure⟩
∈ C.val
```

In the source	Mathematical meaning

Algebra and order
`variable {A : Type u} [NonUnitalCStarAlgebra A] [Nontrivial A]`
`variable [PartialOrder A] [StarOrderedRing A]`
	The nonzero complex C*-algebra A and its compatible order support the pure-state and geometric-point constructions.

Specified component `theorem exists_pureState_kernel_mem_geometricSpectrum (C : NormedSpace.GeometricSpectrum ℂ A) :`	The input $C \in \mathrm{GS}(A)$ is a component subset of $S(A)$, not an individual functional or representation.
Pure state `∃ (phi : StrongDual ℂ A) (hpure : IsPureNonUnitalState A phi),`	There is a bounded complex-linear functional $\varphi : A \to \mathbb{C}$ which is a pure state: positive, of norm one, and extreme among states for real convexity.
Kernel point in that component `extremeKernel ⟨phi, mem_extremePoints_dual_closedBall_of_isPureNonUnitalState phi hpure⟩ ∈ C.val := by`	The extremality result applied to this same φ certifies $\ker\varphi$ as a point of $S(A)$. The full conclusion is $$\ker\varphi \in C.$$ The projection of C is its component subset, so the membership places the kernel point in the originally specified component.

Components as geometric-spectrum points
`def GeometricSpectrum : Type u :=`
`  {C : Set (GeometricStructureSpace 𝕜 E) // C ∈ geometricSpectrumFamily 𝕜 E}`
	A geometric-spectrum point is a component subset of $S(A)$ with its component membership. Its underlying set is the C used in (1).

Exact surrounding source context (separate excerpts)

Exact surrounding source:

```
variable {A : Type u} [NonUnitalCStarAlgebra A] [Nontrivial A]
```

Exact surrounding source:

```
variable [PartialOrder A] [StarOrderedRing A]
```

Related exact source for the reading guide

Algebra and order · Fixed source

```
variable {A : Type u} [NonUnitalCStarAlgebra A] [Nontrivial A]
variable [PartialOrder A] [StarOrderedRing A]
```

Kernel point in that component · Fixed source

```
      extremeKernel ⟨phi, mem_extremePoints_dual_closedBall_of_isPureNonUnitalState phi hpure⟩
∈ C.val := by
```

Components as geometric-spectrum points · Fixed source

```
def GeometricSpectrum : Type u :=
  {C : Set (GeometricStructureSpace 𝕜 E) // C ∈ geometricSpectrumFamily 𝕜 E}
```