

The counterexample with its faithful separable tracial model

`MathlibAnnex.CStarAlgebra.CAR.atomicCounterexampleEndpoint_and_separable_tracial_representation`

Collects the ordinary counterexample, its separable representation, and its faithful unique trace without changing the algebra.

Statement

The fixed CAR-based C^* -algebra A satisfies the complete ordinary counterexample assertion and has a faithful isometric representation ρ on a nonzero separable Hilbert space H_τ . This representation is not irreducible. The distinguished state τ on A is a faithful trace and is the unique tracial state of A .

Assumptions

Let $A \subseteq B(H_{at})$ be the fixed unital C^* -algebra obtained by adjoining the chosen shell-link unitaries to the atomic representation of the CAR algebra C , and then taking the norm-closed unital $*$ -algebra they generate. Let $H_\tau = L^2(C, \tau_C)$ be the GNS Hilbert space of the normalized CAR trace τ_C . Let ρ be the specified representation on H_τ , and let τ be the distinguished extension of the CAR trace to A . No extra assumptions of faithfulness, traciality, simplicity, or CH are made.

Conclusion

The ordinary counterexample assertion holds for this A , including the faithful atomic irreducible model and uniqueness among all nonzero irreducible comparison representations. In addition: H_τ is nontrivial and separable; ρ is injective and isometric; ρ is not nonzero irreducible; τ is a state; $\tau(ab) = \tau(ba)$ for all $a, b \in A$; $\tau(a^*a) = 0$ if and only if $a = 0$; and every tracial state of A equals τ .

Proof route

Assemble the independently proved ordinary, representation, and trace statements for the fixed homogeneity family.

Proof steps

1. Use the complete ordinary assertion and nontriviality and separability of H_τ .
2. Insert injectivity and isometry of ρ and its failure of irreducibility.
3. Insert positivity and normalization of τ , the trace identity, faithfulness, and uniqueness among tracial states.

Lean realization notes

Faithfulness of τ means its explicit square-vanishing criterion and is distinct from injectivity of ρ . The theorem does not infer faithfulness from uniqueness alone. The atomic irreducible model acts on H_{at} , not $H\tau$. The GNS interpretation is supported by the cyclic-vector and pointed-unitary results; the exact conjunction is displayed separately below. This declaration has no assigned level in the pinned Project scope.

Language: en · CARD_CONTENT_COMPLETE

Main citations

[MathlibAnnex.CStarAlgebra.CAR.atomicCounterexampleTrace_star_mul_self_eq_zero_iff](#)

[MathlibAnnex.CStarAlgebra.CAR.eq_atomicCounterexampleTrace_of_mem_stateSpace_of_mul_comm](#)

Lean source declaration (exact)

```
theorem atomicCounterexampleEndpoint_and_separable_tracial_representation :
  AtomicCounterexampleEndpoint.{v}  $\wedge$ 
  Nontrivial SeparableCounterexampleHilbertSpace  $\wedge$ 
  TopologicalSpace.SeparableSpace SeparableCounterexampleHilbertSpace  $\wedge$ 
  Function.Injective separableCounterexampleRepresentation  $\wedge$ 
  Isometry separableCounterexampleRepresentation  $\wedge$ 
   $\neg$  Representation.IsIrreducible separableCounterexampleRepresentation  $\wedge$ 
  atomicCounterexampleTrace  $\in$ 
  MathlibAnnex.Analysis.CStarAlgebra.stateSpace AtomicCounterexampleAlgebra  $\wedge$ 
  ( $\forall$  a b, atomicCounterexampleTrace (a * b) = atomicCounterexampleTrace (b * a))  $\wedge$ 
  ( $\forall$  a, atomicCounterexampleTrace (star a * a) = 0  $\leftrightarrow$  a = 0)  $\wedge$ 
  ( $\forall$   $\phi$  : AtomicCounterexampleAlgebra  $\rightarrow$  L[ $\mathbb{C}$ ]  $\mathbb{C}$ ,
   $\phi \in$  MathlibAnnex.Analysis.CStarAlgebra.stateSpace AtomicCounterexampleAlgebra  $\rightarrow$ 
  ( $\forall$  a b,  $\phi$  (a * b) =  $\phi$  (b * a))  $\rightarrow$   $\phi$  = atomicCounterexampleTrace) :=
  (shellFamilyEndpoint homogeneityShellFamily,
  nontrivial_traceHilbertSpace,
  separableSpace_separableCounterexampleHilbertSpace,
  separableCounterexampleRepresentation_injective,
  isometry_separableCounterexampleRepresentation,
  not_isIrreducible_separableCounterexampleRepresentation,
  atomicCounterexampleTrace_mem_stateSpace,
  atomicCounterexampleTrace_mul_comm,
  atomicCounterexampleTrace_star_mul_self_eq_zero_iff,
  eq_atomicCounterexampleTrace_of_mem_stateSpace_of_mul_comm)
```

Read exact [MathlibAnnex v0.4.0 source](#)

Exact Card identity

Stable Card ID: bd4b1ed05fb5063cf29ec3c970e72b83f91f2924b541f616688ce518163f5dac

Card SHA-256: 4781b2a183efdb30b33f8d73b2423360cd38cdb807c6f9b7ddaa3746770023b2

Approved exposition SHA-256: 9a8b444d9c243ea97df0378601a6864335b6504d32ca2900fc21f8bdcaa6a4f4

AI-assisted material. Selected, edited, and reviewed by Ryotaro Tanaka. The recorded review is source-exposition correspondence, not an independent proof audit.