Compact image of a representative of the unique irreducible-representation class

`MathlibAnnex.Analysis.CStarAlgebra.NonUnitalCStarRepresentation.isCompactOperator_map_of_singleton`

Shows that every operator in the image of a separably acting representative of the unique irreducible-representation class is compact.

Statement

Let A be a nonzero complex C*-algebra, not assumed unital, H a separable complex Hilbert space, and π a representation of A on H that is a representative of the unique unitary-equivalence class of nonzero irreducible representations of A. No simplicity assumption is made. Every operator π(a), a∈A, is compact.

Assumptions

Let A be a nonzero complex C*-algebra, not assumed unital, H a separable complex Hilbert space, and π a representation of A on H that is a representative of the unique unitary-equivalence class of nonzero irreducible representations of A. No simplicity assumption is made.

Conclusion

Every operator π(a), a∈A, is compact.

Proof idea

Assuming a noncompact image, the source takes a noncompact self-adjoint part and embeds it in a maximal abelian unitization subalgebra D. The closed compact-preimage ideal in D does not contain that element; character separation produces χ. The eigenvector associated with a character distinct from the scalar character and a compact rank-one projection contradict χ's annihilation of the ideal. This follows the source's character route, not a norm-separable pure-state plan.

Proof steps

1. Exact Lean statement: ∀ {A : Type u} [inst : NonUnitalCStarAlgebra A] [inst_1 : PartialOrder A] [StarOrderedRing A] {H : Type v} [inst_3 : NormedAddCommGroup H] [inst_4 : InnerProductSpace ℂ H] [inst_5 : CompleteSpace H] [Nontrivial A] [TopologicalSpace.SeparableSpace H] (pi : MathlibAnnex.Analysis.CStarAlgebra.NonUnitalCStarRepresentation A H), pi.IsSingletonIrreducibleModel → ∀ (a : A), IsCompactOperator ⇑(pi a)

2. ∀ a : A, IsCompactOperator (π a).

3. Assuming a noncompact image, the source takes a noncompact self-adjoint part and embeds it in a maximal abelian unitization subalgebra D. The closed compact-preimage ideal in D does not contain that element; character separation produces χ. The eigenvector associated with a character distinct from the scalar character and a compact rank-one projection contradict χ's annihilation of the ideal. This follows the source's character route, not a norm-separable pure-state plan.

Main citations

MathlibAnnex.Analysis.CStarAlgebra.NonUnitalCStarRepresentation.exists_preimage_rankOne_of_singleton

Exact formal dependency; inspect the linked Card and exact source.

MathlibAnnex.Analysis.CStarAlgebra.exists_character_annihilating_closedIdeal_of_not_mem

Exact formal dependency; inspect the linked Card and exact source.

MathlibAnnex.CStarAlgebra.compactPreimageIdeal

Exact formal dependency; inspect the linked Card and exact source.

Lean source signature (exact)

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theorem isCompactOperator_map_of_singleton [Nontrivial A]
[TopologicalSpace.SeparableSpace H]
(pi : NonUnitalCStarRepresentation A H)
(hsingle : IsSingletonIrreducibleModel.{u, v, u} pi) :
∀ a : A, IsCompactOperator (pi a)
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Read exact MathlibAnnex v0.4.0 source

Exact Card identity

Stable Card ID: be93429f16421fb0cf8eb8bb91861473c1fd7b52af3892ee1ff2d10e1fb20c85

Card SHA-256: 9c84c41fbd329d019a48250459bbaa298670bcb68317e4f2c154f1c9e5144ac3

Approved exposition SHA-256: 8c3c5f77b4171d3fa81919e7f7f2167d426edcc10c67c36e31a82665eadfe93c

AI-assisted material. Selected, edited, and reviewed by Ryotaro Tanaka. The recorded review is source–exposition correspondence, not an independent proof audit.