Structural properties of one shell-generated C*-algebra

`MathlibAnnex.CStarAlgebra.CAR.shellFamilyEndpoint`

theorem

Collects the proved properties of a single generated algebra, keeping the shell-family input separate from its consequences.

Statement

For every fixed shell family, form T, ι and ρ_0 as below. Then T is nonzero, norm closed and infinite-dimensional, ι is injective and unital, and ρ_0 is faithful and irreducible. Every nonzero irreducible complex star representation of T, even if not initially required to preserve the unit, is unitarily equivalent to ρ_0. The only norm-closed two-sided ideals of T are 0 and T, and T has no injective star representation whose range is precisely all compact operators on any complex Hilbert space.

Assumptions

Let A be the completed CAR algebra with matrix stages $M_{2^n}(\mathbb{C})$ embedded by $a \mapsto a \otimes I_2$. Write f_n for the image of the first diagonal matrix unit, $p_n = f_n - f_{n+1}$, and ω for the root state characterized by $\omega(\iota_n(a)) = a_{00}$ on each stage, where ι_n is its canonical embedding.

Index chosen pure-state Gelfand-Naimark-Segal (GNS) representations by their unitary-equivalence classes $j \in I$, with root index o and representative states φ_j.

A fixed shell family consists of complex-linear unital star automorphisms α_j and elements $w_{j,n} \in A$ satisfying $\varphi_j(\alpha_j(a)) = \omega(a)$, $w_{j,n}^* w_{j,n} = \alpha_j(p_n)$ and $w_{j,n} w_{j,n}^* = p_n$.

At the root, α_o is the identity and $w_{o,n} = p_n$.

Let $\mathcal{H} = \bigoplus_{j \in I} H_j$ be the Hilbert direct sum of these GNS spaces, $\pi = \bigoplus_j \pi_j$ their representation of A, and $b_j = e_j \xi_j \in \mathcal{H}$ its embedded unit cyclic vector, where $\xi_j \in H_j$ is the selected unit cyclic vector and $e_j : H_j \to \mathcal{H}$ is coordinate inclusion.

Choose once a family of unitary operators $L_j \in B(\mathcal{H})$ from the construction of the shell unitaries: $L_j b_j = b_o$, $L_j \pi(\alpha_j(p_n)) = \pi(w_{j,n})$, and $L_o = I_{\mathcal{H}}$. Here $B(\mathcal{H})$ denotes the bounded complex-linear operators on $\mathcal{H}$.

Put $T = C^*(\pi(A), \{L_j : j \in I\}) \subseteq B(\mathcal{H})$, the norm-closed unital star algebra generated by these operators. Let $\iota : A \to T$ be $\iota(a) = \pi(a)$ with its codomain restricted to T, and let $\rho_0 : T \to B(\mathcal{H})$ be inclusion.

The comparison Hilbert space K may be any complete complex Hilbert space in an arbitrary independent universe. No separability or dimension bound on K is added. The chosen family, links and algebra T do not depend on that comparison universe.

Conclusion

All conclusions concern this same concrete T. In the capture assertion, unitary equivalence means there is a surjective complex-linear isometry $U : \mathcal{H} \to K$ with $U(\rho_0(t)x) = \rho(t)(Ux)$ for all $t \in T$ and $x \in \mathcal{H}$. In the compact-operator assertion, the competing map need not be unital and the zero Hilbert space is also covered.

Proof route

The construction of the shell unitaries gives a faithful source map and irreducible inclusion. The unital copy of the infinite-dimensional CAR algebra makes T nonzero and infinite-dimensional. The universal capture theorem supplies unitary equivalence for unital irreducible representations; irreducibility forces any nonzero possibly nonunital representation to preserve the unit. Ideal-separating pure GNS representations then yield simplicity. Finally a unital infinite-dimensional algebra cannot be represented injectively onto all compact operators.

Proof steps

1. Choose the links once, restrict π to the generated algebra, and use its faithful root summand. Norm closure is part of the construction of T.

2. If T were finite-dimensional, its injective complex-linear source map would make A finite-dimensional, contradicting the growing matrix stages.

3. Apply the unitary-equivalence theorem, the nonunital-to-unital argument, and the closed-ideal consequence. Exclude an isomorphism of this same infinite-dimensional algebra onto all compact operators. These are separate proved results supplying the listed fields.

Main citations

- Exact declaration and proof · Exact source
- MathlibAnnex.CStarAlgebra.CAR.ShellFamilyEndpoint · Exact source
- MathlibAnnex.CStarAlgebra.CAR.RepresentativeShellFamily · Exact source
- MathlibAnnex.CStarAlgebra.CAR.exists_completedAtomicShellModel · Exact source
- MathlibAnnex.CStarAlgebra.CAR.shellFamilyLinks · Exact source
- MathlibAnnex.CStarAlgebra.CAR.ShellFamilyTarget · Exact source
- MathlibAnnex.CStarAlgebra.CAR.shellFamilySourceHom · Exact source
- MathlibAnnex.CStarAlgebra.CAR.shellFamilyInclusion · Exact source
- MathlibAnnex.CStarAlgebra.CAR.not_finiteDimensional_shellFamilyTarget · Exact source
- MathlibAnnex.CStarAlgebra.CAR.isUniqueIrreducibleModel_shellFamilyInclusion · Exact source
- MathlibAnnex.CStarAlgebra.CAR.shellFamilyInclusion_unitaryEquivalent_nonUnital · Exact source
- MathlibAnnex.CStarAlgebra.CAR.isSimpleCStarAlgebra_shellFamilyTarget · Exact source
- MathlibAnnex.CStarAlgebra.CAR.not_isCompactOperatorModel_shellFamilyTarget · Exact source

Lean source signature (exact)

```
theorem shellFamilyEndpoint (family : RepresentativeShellFamily) :
    ShellFamilyEndpoint.{v} family
```

> Here `ShellFamilyTarget family` is T, `shellFamilySourceHom family` is ι, and `shellFamilyInclusion family` is ρ_0. `ShellFamilyEndpoint` is the proposition listing the properties in the Statement; the theorem `shellFamilyEndpoint` proves it. Its exact structure declaration is linked as “Properties of the fixed target” above. The family, and hence T, are fixed before the comparison Hilbert space is chosen.

Lean realization notes

The separately linked Lean structure lists these properties; defining that proposition does not prove it. This theorem proves that proposition for every supplied family. The family hypothesis remains explicit. The structural theorem for the homogeneity-based algebra and its faithful representation on a separable Hilbert space specialize this fixed-family result.

▼ **Exact Card identity**

Language: en
CARD_CONTENT_COMPLETE
Card UID: lfh:lfh-declaration-card:sha256:c2533b47d2800cf40bedcc7a2969c4296ab1b60fe6986fe527e1247ef3862788
Card revision: 1 · SHA-256: 963ba28055994c54f1f2a45c157c1d4c74f37952b595c5894c0c051169f78338
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