Pointed unitary transport between trace-cyclic target representations

`MathlibAnnex.CStarAlgebra.CAR.exists_pointed_unitary_of_trace_of_cyclic`

theorem

Source cyclicity and strong shell sums extend the canonical CAR orbit isometry to the whole target.

Statement

Let A be the completed CAR algebra, the completion of the matrix system $M_{2^n}(\mathbb{C})$, and let τ be its normalized trace. Fix a representative shell family: for each chosen pure-GNS class j it specifies an automorphism α_j and partial isometries $w_{j,n}$ between the transported and root shells. Fix the links L_j chosen by this family in the selected atomic representation π. Throughout this Card,

$$T = C^*(\pi(A), \{L_j\}), \qquad \iota : A \longrightarrow T, \quad \iota(a) = \pi(a).$$

Thus T is the already constructed concrete unital C*-algebra, and ι is its injective source map. The links and target are kept fixed. A state means a positive continuous complex linear functional taking 1 to 1. Let $\rho : T \to B(H)$ and $\sigma : T \to B(K)$ be unital star representations on complex Hilbert spaces. Suppose $\xi \in H$ and $\eta \in K$ have dense target orbits and realize the same source trace:

$$\langle \xi, \rho(\iota(a))\xi \rangle = \tau(a) = \langle \eta, \sigma(\iota(a))\eta \rangle \quad (a \in A).$$

There exists a unitary $E : H \to K$ with $E\xi = \eta$ and $E\rho(t) = \sigma(t)E$ for every $t \in T$.

Assumptions

Both Hilbert spaces are complete. The source trace identities and density of the two target orbits are the hypotheses. The vectors automatically have norm 1 by taking $a = 1$. No irreducibility of either representation and no norm separability of T is required.

Conclusion

The resulting unitary is pointed and intertwines the entire fixed target. Source intertwining is obtained first; intertwining the added generators is a separate strong-limit step.

Proof route

Trace estimates kill the residual flag projections on the source-cyclic subspace. Reduction and target cyclicity make this subspace the entire Hilbert space. Pointed CAR transport can then be passed through the shell limits and extended by norm generation.

Proof steps

1. Write f_n for the decreasing root projections, $p_n = f_n - f_{n+1}$, and α_j for the automorphism selected for class j. The shell operators satisfy $w_{j,n}^* w_{j,n} = \alpha_j(p_n)$ and $w_{j,n} w_{j,n}^* = p_n$. The normalized source trace has $\tau(f_n) = 2^{-n}$. Traciality on these supports gives

$\tau(\alpha_j(f_n)) - \tau(\alpha_j(f_{n+1})) = \tau(p_n).$

Together with $f_0 = \alpha_j(f_0) = 1$, this yields $\tau(\alpha_j(f_n)) = 2^{-n}$, as recorded by the transported-flag trace theorem.

2. Put $M = \overline{\rho(\iota(A))\xi}$. For every fixed $b \in A$, apply the projection-orbit estimate separately to the root projection f_n and the transported projection $\alpha_j(f_n)$:

$$\|\rho(\iota(f_n))\rho(\iota(b))\xi\|^2 \le \|b\|^2\tau(f_n) = \|b\|^2 2^{-n},$$
$$\|\rho(\iota(\alpha_j(f_n)))\rho(\iota(b))\xi\|^2 \le \|b\|^2\tau(\alpha_j(f_n)) = \|b\|^2 2^{-n}.$$

Both right-hand sides tend to zero as $n \to \infty$. These estimates use the tracial source vector state τ; no traciality of a state on T is used.

3. Actual-target shell reconstruction provides strong limits S_j, S_j^* of the represented shell and adjoint-shell sums, intersection projections P_j, Q_j for the transported and root flags, and a residual operator R_j, with

$$\rho(g_j) = S_j + R_j, \qquad R_j = \rho(g_j)P_j = Q_jR_jP_j,$$

where $g_j = L_j \in T$. For every n, the intersection projection satisfies $P_j\rho(\iota(\alpha_j(f_n))) = P_j$. Hence, for every fixed $b \in A$,

$$\begin{aligned}\|P_j\rho(\iota(b))\xi\| &= \|P_j\rho(\iota(\alpha_j(f_n)))\rho(\iota(b))\xi\|\\ &\le \|\rho(\iota(\alpha_j(f_n)))\rho(\iota(b))\xi\| \longrightarrow 0.\end{aligned}$$

The left-hand side is independent of n, so $P_j\rho(\iota(b))\xi = 0$. Using $Q_j\rho(\iota(f_n)) = Q_j$ and the root-projection estimate gives $Q_j\rho(\iota(b))\xi = 0$ as well. Continuity extends both equalities to M, so $P_j|_M = Q_j|_M = 0$. Therefore $R_j|_M = 0$; taking adjoints of the support relation gives $R_j^* = P_jR_j^*Q_j$ and $R_j^*|_M = 0$.

4. The closed subspace M reduces the source action. Both the shell sums and their adjoints preserve it, so their strong limits preserve it as well. The preceding residual equalities show $\rho(g_j)|_M = S_j|_M$ and $\rho(g_j)^*|_M = S_j^*|_M$. Hence M reduces all generators, and therefore the norm-closed algebra they generate. It contains ξ and thus the entire target orbit. Target cyclicity forces $M = H$. The same argument gives $\overline{\sigma(\iota(A))\eta} = K$. These are the conclusions of `reduces_cyclicSubspace_of_trace` and `denseRange_source_orbit_of_trace_of_cyclic`.

5. For $a, b \in A$, equality of source vector states gives the Gram identity

$$\langle\rho(\iota(a))\xi, \rho(\iota(b))\xi\rangle = \tau(a^*b) = \langle\sigma(\iota(a))\eta, \sigma(\iota(b))\eta\rangle.$$

The pointed cyclic transport theorem applies to the two now-dense source orbits. The orbit isometry extends onto K and yields a unitary $E : H \to K$ such that $E\xi = \eta$ and $E\rho(\iota(a)) = \sigma(\iota(a))E$ for every $a \in A$.

6. Set $s_{j,N} = \sum_{n<N} w_{j,n}$. Since the source-cyclic spaces are the whole spaces, reconstruction gives, in each representation,

$$\rho(\iota(s_{j,N})) \longrightarrow \rho(g_j), \qquad \sigma(\iota(s_{j,N})) \longrightarrow \sigma(g_j)$$

strongly, as well as the corresponding adjoint limits. For $x \in H$, source intertwining at every finite sum reads

$$E\rho(\iota(s_{j,N}))x = \sigma(\iota(s_{j,N}))Ex.$$

Continuity of E and the two displayed limits yield $E\rho(g_j)x = \sigma(g_j)Ex$. This is the limit passage in the linked proof of the stated theorem.

7. Conjugation gives two continuous star homomorphisms $E\rho(\cdot)E^{-1}$ and σ on the same concrete T. They agree on $\iota(A)$ and every g_j, hence on the algebraic star algebra generated by them and then its norm closure. The exact supplier `intertwines_of_source_of_generators` yields $E\rho(t) = \sigma(t)E$ for all $t \in T$.

Main citations

- The stated existence or structural result · Exact source
- The fixed concrete target and source map · Exact source
- Faithfulness of the source embedding · Exact source
- Trace values on root and transported flags · Exact source
- Projection-orbit estimate for a tracial state · Exact source
- Intersection projections vanish on the source-cyclic space · Exact source
- Actual-representation shell reconstruction · Exact source
- Shell and adjoint sums on the source-cyclic space · Exact source
- Reduction by the source-cyclic space · Exact source
- Target cyclicity makes the source-cyclic space full · Exact source
- Dense source orbit · Exact source
- Strong sums in each trace-cyclic representation · Exact source
- Pointed cyclic transport for the common source state · Exact source
- Intertwining extends from concrete generators · Exact source

Lean source signature (exact)

```
theorem exists_pointed_unitary_of_trace_of_cyclic
    (family : RepresentativeShellFamily)
    (ρ : Representation (ShellFamilyTarget family) H)
    (σ : Representation (ShellFamilyTarget family) K) (ξ : H) (η : K)
    (hξ : ∀ a, Representation.vectorFunctional
      (ρ.comp (shellFamilySourceHom family)) ξ a = trace a)
    (hη : ∀ a, Representation.vectorFunctional
      (σ.comp (shellFamilySourceHom family)) η a = trace a)
    (hρ : DenseRange (fun a ↦ ρ a ξ)) (hσ : DenseRange (fun a ↦ σ a η)) :
    ∃ e : H ≃ₗᵢ[ℂ] K, e ξ = η ∧
      ∀ a, (e : H →L[ℂ] K).comp (ρ a) = (σ a).comp (e : H →L[ℂ] K)
```

> `family` fixes T and ι. The two `Representation` arguments are ρ, σ and their vectors are ξ, η. `hξ` and `hη` are equality of the source vector functionals with `trace`; The arguments `hρ` and `hσ` concern target orbits. The returned `e` points from the first Hilbert space to the second. The linked proof's source restrictions, source-density facts and shell-sum limits are proof-local names; the displayed signature retains the full quantified conclusion. In continuous-operator source, `†` is the adjoint written * in the mathematics.

Lean realization notes

All shell convergence used below is proved inside each actual representation. The argument does not pass strong convergence through an arbitrary star homomorphism. A star on a bounded operator denotes its Hilbert adjoint, corresponding to Lean's postfix dagger on continuous linear operators.

▼ **Exact Card identity**

Language: en

CARD_CONTENT_COMPLETE

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