A represented rank-one projection from a representative of the unique irreducible-representation class

`MathlibAnnex.Analysis.CStarAlgebra.NonUnitalCStarRepresentation.exists_nonzero_projection_rankOne_map`

Produces a nonzero projection whose represented image is a compact rank-one orthogonal projection.

Statement

Let A be a nonzero complex C*-algebra, not assumed unital, H a separable complex Hilbert space, and π a representation of A on H that is a representative of the unique unitary-equivalence class of nonzero irreducible representations of A. There are a nonzero star projection p∈A and a unit vector e∈H with π(p)=rankOne e e; the represented operator π(p) is compact. Minimality is not an explicit clause of this theorem type.

Assumptions

Let A be a nonzero complex C*-algebra, not assumed unital, H a separable complex Hilbert space, and π a representation of A on H that is a representative of the unique unitary-equivalence class of nonzero irreducible representations of A.

Conclusion

There are a nonzero star projection p∈A and a unit vector e∈H with π(p)=rankOne e e; the represented operator π(p) is compact. Minimality is not an explicit clause of this theorem type.

Proof idea

The preceding scalar-corner theorem gives p. Injectivity makes π(p) nonzero, and the scalar-corner/rank-one lemma gives a unit e and the image equation. Compactness follows from compactness of a rank-one operator.

Proof steps

1. Exact Lean statement: ∀ {A : Type u} [inst : NonUnitalCStarAlgebra A] {H : Type v} [inst_1 : NormedAddCommGroup H] [inst_2 : InnerProductSpace ℂ H] [inst_3 : CompleteSpace H] [Nontrivial A] [inst_5 : PartialOrder A] [StarOrderedRing A] [TopologicalSpace.SeparableSpace H] (pi : MathlibAnnex.Analysis.CStarAlgebra.NonUnitalCStarRepresentation A H), pi.IsSingletonIrreducibleModel → ∃ p, IsStarProjection p ∧ p ≠ 0 ∧ (∃ e, ‖e‖ = 1 ∧ pi p = ((InnerProductSpace.rankOne ℂ) e) e) ∧ IsCompactOperator ⇑(pi p)

2. ∃ p, IsStarProjection p ∧ p≠0 ∧ (∃ e, ‖e‖=1 ∧ π p=rankOne ℂ e e) ∧ IsCompactOperator (π p).

3. The preceding scalar-corner theorem gives p. Injectivity makes π(p) nonzero, and the scalar-corner/rank-one lemma gives a unit e and the image equation. Compactness follows from compactness of a rank-one operator.

Main citations

MathlibAnnex.Analysis.CStarAlgebra.NonUnitalCStarRepresentation.exists_nonzero_projection_scalar_corner

Exact formal dependency; inspect the linked Card and exact source.

MathlibAnnex.Analysis.CStarAlgebra.NonUnitalCStarRepresentation.injective_of_singleton

Exact formal dependency; inspect the linked Card and exact source.

Lean source signature (exact)

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theorem exists_nonzero_projection_rankOne_map [Nontrivial A]
[PartialOrder A] [StarOrderedRing A]
[TopologicalSpace.SeparableSpace H]
(pi : NonUnitalCStarRepresentation A H)
(hsingle : IsSingletonIrreducibleModel.{u, v, u} pi) :
∃ p : A, IsStarProjection p ∧ p ≠ 0 ∧
(∃ e : H, ‖e‖ = 1 ∧ pi p = InnerProductSpace.rankOne ℂ e e) ∧
IsCompactOperator (pi p)
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Read exact MathlibAnnex v0.4.0 source

Exact Card identity

Stable Card ID: c4c4491fa52a7fbdbfde28f3dd9f46756ece7e43d6d9f8b3afb346db83d351e4

Card SHA-256: 0f8e7968c7269244dff3347b8dd04824dd6f183f487dc52919fbc4e7b19ca3c9

Approved exposition SHA-256: 51b034cb324b0f53556680512dc0f125c6e75dc8825eb8940bf66089d36dd418

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