Semilinear vector changes preserve finite normal-part closures

`MathlibAnnex.Analysis.CStarAlgebra.normalPartSemilinearFiniteHomeomorph`

def

Defines a finite homeomorphism of normal parts from two semilinear vector isomorphisms.

Statement

Let $A_1 \subseteq \mathcal{B}(H)$ and $A_2 \subseteq \mathcal{B}(K)$ be norm-closed complex $*$-subalgebras with nonzero irreducible inclusion representations on complete complex Hilbert spaces. Neither algebra needs an identity. Use inner products linear in their second variable. For nonzero vectors, write

$$f_{x,y}^{A_1}(a) = \langle y, a(x)\rangle \quad (a \in A_1), \qquad f_{z,w}^{A_2}(b) = \langle w, b(z)\rangle \quad (b \in A_2).$$

Let $N(A_j) \subseteq S(A_j)$ be the coefficient-kernel normal part, with inherited closure

$$c_{A_j}(D) = \left\{ I \in N(A_j) : \bigcap_{J \in D} J \subseteq I \right\}.$$

Suppose $\sigma : \mathbb{C} \to \mathbb{C}$ is a unital ring homomorphism with $\sigma^2 = \mathrm{id}$ and $\sigma(\overline{c}) = \overline{\sigma(c)}$ for every c, and $U, V : H \to K$ are algebraic σ-semilinear isomorphisms. The definition equips

$$F : N(A_1) \longrightarrow N(A_2), \qquad F(\ker f_{x,y}^{A_1}) = \ker f_{Ux,Vy}^{A_2} \tag{1}$$

with finite continuity in both directions. Thus F is a finite-homeomorphism: a bijection whose map and inverse preserve membership in closures of finite sets.

Definition

The underlying equivalence sends the coefficient kernel represented by (x, y) to the kernel represented by (Ux, Vy). Its inverse is

$$F^{-1} : N(A_2) \longrightarrow N(A_1), \qquad F^{-1}(\ker f_{z,w}^{A_2}) = \ker f_{U^{-1}z,V^{-1}w}^{A_1}. \tag{2}$$

The semilinear kernel bijection has the forward evaluation for every nonzero representing pair. Its inverse uses the inverses of those same U, V.

The other two fields are the finite-continuity requirements

$$\begin{aligned} F(c_{A_1}(D)) &\subseteq c_{A_2}(F(D)) && (D \subseteq N(A_1) \text{ finite}), \\ F^{-1}(c_{A_2}(E)) &\subseteq c_{A_1}(F^{-1}(E)) && (E \subseteq N(A_2) \text{ finite}). \end{aligned} \tag{3}$$

The finite-continuity theorem guarantees these inclusions for U, V and their inverses, respectively. Both inverse maps use the same self-inverse scalar homomorphism σ.

Assumptions

The same scalar homomorphism governs both vector maps. Its inverse is itself, and it commutes with complex conjugation. U, V are additive bijections with $U(cx) = \sigma(c)Ux$ and $V(cy) = \sigma(c)Vy$. They are not assumed bounded, isometric, unitary, or algebra intertwiners. Closedness and irreducibility belong to the two concrete algebras; they guarantee the coefficient-kernel realization and the finite-continuity theorem.

Conclusion

The output is a bijection of the normal parts with finite continuity of the forward map and its inverse, for all finite subsets including the empty set. Full continuity for arbitrary subsets is not a field of this construction.

Main citations

- Exact finite-homeomorphism construction: `normalPartSemilinearFiniteHomeomorph` and its three defining fields.
- arXiv:2504.05551v1, Theorem 6.9: the algebraic finite-homeomorphism layer, separated from its bounded full-continuity layer.
- normalPartSemilinearMap_point: representative-independent evaluation.
- normalPartSemilinearEquiv: the specified bijection.
- finitelyContinuous_normalPartSemilinearMap: finite continuity.

Lean source signature (exact)

The complete definition below is copied from the pinned source, including its defining right-hand side.

```
def normalPartSemilinearFiniteHomeomorph
    (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype S))
    (hT : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype T))
    (hσ : ∀ c : ℂ, σ (star c) = star (σ c)) (U V : H ≃ₛₗ[σ] K) :
    ClosureSpace.FiniteHomeomorph
      ((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S))
      ((NormedSpace.geometricStructureClosure ℂ T).subspace (normalPart T)) where
  toEquiv := normalPartSemilinearEquiv S T hS hT hσ U V
  finitelyContinuous_toFun := finitelyContinuous_normalPartSemilinearMap S T hS hT hσ U V
  finitelyContinuous_invFun :=
    finitelyContinuous_normalPartSemilinearMap T S hT hS hσ U.symm V.symm
```

In the source, S and T denote the algebras A_1 and A_2 used here, respectively.

In the source	Mathematical meaning

First algebra and Hilbert space
`variable {H : Type u} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]`
`variable (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H))`
`variable [IsClosed (S : Set (H →L[ℂ] H))]`
	The complex Hilbert space H and the norm-closed complex $*$-subalgebra $A_1 \subseteq \mathcal{B}(H)$. The source calls this algebra S.

Second algebra and scalar map
`variable {K : Type v} [NormedAddCommGroup K] [InnerProductSpace ℂ K] [CompleteSpace K]`
`variable (T : NonUnitalStarSubalgebra ℂ (K →L[ℂ] K))`
`variable [IsClosed (T : Set (K →L[ℂ] K))]`
`variable {σ : ℂ →+* ℂ} [RingHomInvPair σ σ]`
	The complex Hilbert space K and the norm-closed algebra $A_2 \subseteq \mathcal{B}(K)$, called T. The unital scalar homomorphism $\sigma : \mathbb{C} \to \mathbb{C}$ satisfies $\sigma \circ \sigma = \mathrm{id}$; the inverse vector maps therefore use the same scalar map.

Irreducible actions and semilinear maps
`(hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype S))`
`(hT : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype T))`
`(hσ : ∀ c : ℂ, σ (star c) = star (σ c)) (U V : H ≃ₛₗ[σ] K) :`
	The two algebras act nontrivially and irreducibly on their respective spaces. The maps $U, V : H \to K$ are additive bijections satisfying $U(cx) = \sigma(c)U(x)$ and $V(cx) = \sigma(c)V(x)$. The scalar condition is $\sigma(\bar{c}) = \overline{\sigma(c)}$ for every $c \in \mathbb{C}$. No boundedness is assumed.

Domain and target closures
`ClosureSpace.FiniteHomeomorph`
`  ((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S))`
`  ((NormedSpace.geometricStructureClosure ℂ T).subspace (normalPart T)) where`
	The output is a finite homeomorphism $F : N(A_1) \to N(A_2)$. Each `subspace` restricts the geometric closure to the indicated normal part; these are the relative closures c_{A_1}, c_{A_2} used in (3).

Underlying bijection `toEquiv := normalPartSemilinearEquiv S T hS hT hσ U V`	This field uses the bijection induced by the same U, V: $$F(\ker f_{x,y}^{A_1}) = \ker f_{Ux,Vy}^{A_2}.$$ Here $x, y \in H \setminus \{0\}$. Its inverse is the map in (2).

Forward and inverse maps
`def normalPartSemilinearEquiv`
`    (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype S))`
`    (hT : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype T))`
`    (hσ : ∀ c : ℂ, σ (star c) = star (σ c)) (U V : H ≃ₛₗ[σ] K) :`
`    normalPart S ≃ normalPart T where`
`  toFun := normalPartSemilinearMap S T hT U V`
`  invFun := normalPartSemilinearMap T S hS U.symm V.symm`
	The equivalence has forward map $F : N(A_1) \to N(A_2)$ and inverse $F^{-1} : N(A_2) \to N(A_1)$. The inverse uses U^{-1}, V^{-1}, so for $z, w \in K \setminus \{0\}$,
	$$F^{-1}(\ker f_{z,w}^{A_2}) = \ker f_{U^{-1}z,V^{-1}w}^{A_1}.$$
	This excerpt specifies the two maps; the inverse-law proof fields remain in the full source.

In the source	Mathematical meaning
Finite forward continuity `finitelyContinuous_toFun := finitelyContinuous_normalPartSemilinearMap S T hS hT hσ U V`	The cited result supplies, for every finite $D \subseteq N(A_1)$, $$F(c_{A_1}(D)) \subseteq c_{A_2}(F(D)).$$ It applies to the same U, V and scalar condition.
Finite inverse continuity `finitelyContinuous_invFun := finitelyContinuous_normalPartSemilinearMap T S hT hS hσ U.symm V.symm`	The algebras and irreducibility hypotheses are exchanged, and `symm` takes the actual inverses of U, V. For every finite $E \subseteq N(A_2)$, $$F^{-1}(c_{A_2}(E)) \subseteq c_{A_1}(F^{-1}(E)).$$ Neither field asserts continuity on arbitrary subsets.

The declaration retains the full three-field right-hand side. The self-inverse scalar condition is in the separately displayed ambient context.

Exact surrounding source context (separate excerpts)

Exact surrounding source:

```
variable {H : Type u} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
variable (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H))
variable [IsClosed (S : Set (H →L[ℂ] H))]
```

Exact surrounding source:

```
variable {K : Type v} [NormedAddCommGroup K] [InnerProductSpace ℂ K] [CompleteSpace K]
variable (T : NonUnitalStarSubalgebra ℂ (K →L[ℂ] K))
variable [IsClosed (T : Set (K →L[ℂ] K))]
variable {σ : ℂ →+* ℂ} [RingHomInvPair σ σ]
```

Related exact source for the reading guide

Forward and inverse maps · Fixed source

```
def normalPartSemilinearEquiv
    (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype S))
    (hT : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype T))
    (hσ : ∀ c : ℂ, σ (star c) = star (σ c)) (U V : H ≃ₛₗ[σ] K) :
    normalPart S ≃ normalPart T where
  toFun := normalPartSemilinearMap S T hT U V
  invFun := normalPartSemilinearMap T S hS U.symm V.symm
```