

Choosing the CAR shell family from proved homogeneity

MathlibAnnex.CStarAlgebra.CAR.homogeneityShellFamily

def

CAR homogeneity and exact shell matching supply one family whose distinguished component is fixed by an explicit identity choice.

Statement

Let A be the completed CAR algebra, the norm completion of the matrix stages $A_n = M_{2^n}(\mathbb{C})$ under $a \mapsto a \otimes I_2$, with the fixed coordinate reindexing. Write $\iota_n : A_n \rightarrow A$ for the canonical inclusion, $\omega(\iota_n(a)) = a_{00}$ for the root state, $f_n = \iota_n(E_{00})$ for its decreasing root projections, and $p_n = f_n - f_{n+1}$ for the difference shells. Each p_n is a self-adjoint projection. Let I be the set of unitary-equivalence classes of pure-state Gelfand–Naimark–Segal (GNS) representations of A . Choose a representative pure state φ_j for each $j \in I$, with the distinguished class $o = [\omega]$ represented by $\varphi_o = \omega$. Fix the following representative shell family. At o , choose $\alpha_o = \text{id}_A$ and $w_{o,n} = p_n$. At every other j , choose an automorphism α_j from CAR homogeneity for the pair (φ_j, ω) , and then choose shell links for that same automorphism. Thus $\varphi_j(\alpha_j(a)) = \omega(a)$, $w_{j,n}^* w_{j,n} = \alpha_j(p_n)$, and $w_{j,n} w_{j,n}^* = p_n$. This fixed choice is the homogeneity shell family.

Definition

The component supplier branches on $j = o$. On the root branch, the root-state representative is exactly ω and $p_n^* p_n = p_n = p_n p_n^*$, so identity and the shells themselves satisfy all component equations. On a non-root branch, homogeneity gives one α_j with $\varphi_j \circ \alpha_j = \omega$ and approximate innerness. Apply the arbitrary-family shell-matching theorem to this α_j and $(p_n)_n$, and choose its link sequence. The two root lemmas certify the explicit branch, and the displayed definition assembles these data into one family.

Assumptions

The selected states are pure, including the root state. CAR homogeneity is a proved result, and the difference shells are projections. No external KOS proposition or pre-existing shell family is an input to this definition.

Conclusion

The definition supplies a value of the representative-family structure, including both root equations. The same selected value can be carried into later constructions without making a fresh choice at each use.

Main citations

- Definition and its exact construction · Exact source
- Component choice and its explicit root branch · Exact source
- CAR state transport supplying each non-root automorphism · Exact source
- Links for all shells of that fixed automorphism · Exact source
- Identity at the distinguished class · Exact source
- Shells themselves at the distinguished class · Exact source
- The family interface with all fields · Exact source

Lean source signature (exact)

```
noncomputable def homogeneityShellFamily : RepresentativeShellFamily where
  data := representativeShellDataOfHomogeneity
  root_alpha := representativeShellDataOfHomogeneity_root_alpha
  root_link := representativeShellDataOfHomogeneity_root_link
```

Here `data := representativeShellDataOfHomogeneity` is the component choice just described. The two named root lemmas provide the `root_alpha` and `root_link` fields. `homogeneityShellFamily` is a fixed value, not a new existentially chosen family each time it is referenced. The component supplier is linked separately and is not concatenated into this definition's signature.

Lean realization notes

This is a fixed choice, not a uniqueness theorem for all possible families. In particular, the non-root links need not be unique. The subsequent structural theorem uses this very family. The present definition does not itself prove uniqueness of the irreducible representation class, the rank-one identities, simplicity, or the exclusion of compact-operator algebras.

▼ Exact Card identity

Language: en

CARD_CONTENT_COMPLETE

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