# Irreducible dimension at most a fixed number

`MathlibAnnex.Analysis.CStarAlgebra.forall_irreducible_finrank_le_iff_component_compact_homeomorph`

**theorem**

Characterizes a uniform finite dimension bound by full compact component models.

## Statement

Let $A$ be a nonzero complex C*-algebra, and let $n \in \mathbb{N}$.

For a complex C*-algebra $D$, write $S(D)$ for its geometric structure space. Its points are the unions of kernels of norm-one bounded complex-linear functionals supporting a maximal proper face of the closed unit ball of $D$; faces use real convexity. Write $c_D$ for the hull–kernel closure on $S(D)$:

$$c_D(E) = \left\{ I \in S(D) : \bigcap_{J \in E} J \subseteq I \right\}, \qquad E \subseteq S(D).$$

These are the geometric structure definition and its closure, with the norm-one description justified by the norm-one support characterization.

For $I \in S(A)$, let $C_A(I)$ be its finite-chain component: the points reachable from $I$ by a finite chain in which adjacent points are equal or form a nonclosed two-point set. Give $C_A(I)$ the induced closure

$$c_{C_A(I)}(E) = C_A(I) \cap c_A(E), \qquad E \subseteq C_A(I).$$

For a complex Hilbert space $H$, let $\mathcal{K}(H)$ be the algebra of compact operators on $H$. In particular, $c_{\mathcal{K}(K)}$ denotes closure on $S(\mathcal{K}(K))$, where $\mathcal{K}(K)$ is itself the algebra to which the geometric structure construction is applied.

The following conditions are equivalent:

- Every irreducible $*$-representation $\pi : A \to \mathcal{B}(H)$ on a complex Hilbert space has $H$ finite-dimensional and $\dim_{\mathbb{C}} H \le n$.
- For every $I \in S(A)$, there are a nonzero finite-dimensional complex Hilbert space $K$ with $\dim_{\mathbb{C}} K \le n$ and a bijection $e : C_A(I) \to S(\mathcal{K}(K))$ such that

$$e(c_{C_A(I)}(E)) = c_{\mathcal{K}(K)}(e(E)) \qquad (E \subseteq C_A(I)). \quad (1)$$

Here $c_{\mathcal{K}(K)}$ is the closure on the geometric structure space of the compact-operator algebra on this same $K$. Thus $e$ is a homeomorphism of closure spaces. The model $K$ and the map $e$ may depend on $I$; the bound $n$ is common to all components.

## Assumptions

The algebra $A$ is nonzero; no assumption $n > 0$ is made. Representations are $*$-homomorphisms into bounded operators, without a requirement of unitality. Irreducibility includes a nonzero action, but does not require faithfulness. All Hilbert spaces are complex and complete. The compatible order on $A$ supplies positivity and pure states. Finite-dimensionality is a conclusion on the representation side and a requirement on the model side. It is separate from the numerical dimension bound. No separability hypothesis is added.

## Conclusion

A uniform finite bound on the dimensions of all irreducible representations is equivalent to the compact-operator component models in (1), with the same bound. For nonzero $A$, both sides are false at $n = 0$, by the cited zero-bound exclusions. This does not add $n > 0$ as an input assumption.

## Main citations

The bounded-dimension characterization specializes the general dimension-predicate theorem to the upper bound $n$. The finite-dimensional compact-model predicate keeps finiteness and the bound on the same target. The same-map full-continuity theorem preserves the finite comparison and its inverse. The representation-side zero-bound exclusion and component-side zero-bound exclusion cover $n = 0$.

## Lean source signature (exact)

Exact source and declaration. Only the literal signature is displayed here; the linked source retains the proof.

```
theorem forall_irreducible_finrank_le_iff_component_compact_homeomorph (n : ℕ) :
    (∀ (L : Type u) [NormedAddCommGroup L] [InnerProductSpace ℂ L] [CompleteSpace L],
      ∀ pi : NonUnitalCStarRepresentation A L, pi.IsIrreducible →
        FiniteDimensional ℂ L ∧ Module.finrank ℂ L ≤ n) ↔
    ∀ I : NormedSpace.GeometricStructureSpace ℂ A,
      ClosureSpace.HasFiniteDimensionalCompactHomeomorph.{u, u}
        ((NormedSpace.geometricStructureClosure ℂ A).subspace
          ((NormedSpace.geometricStructureClosure ℂ A).component I)) (fun m => m ≤ n)
```

The principal equivalence fixes the universe of $A$ and quantifies both representation spaces and compact-model spaces in that same universe. The dimension transport to an independently specified Hilbert space is a separate theorem, not a quantifier over all universes in this equivalence.

| In the source | Mathematical meaning |
| --- | --- |
| **Ambient algebra and compatible order**<br>`universe u w`<br>`variable {A : Type u} [NonUnitalCStarAlgebra A] [hA : Nontrivial A]`<br>`attribute [local instance] ContinuousLinearMap.isClosed_compactOperatorStarSubalgebra`<br>`  _root_.CStarAlgebra.spectralOrder _root_.CStarAlgebra.spectralOrderedRing` | The nonzero complex C*-algebra $A$ is fixed. The local instances supply its compatible order and closedness of the compact-operator algebra. The universe of representation and model spaces is the universe of $A$. |
| **Effective nonzero hypothesis**<br>`include hA in` | The preceding nontriviality instance for $A$ is included as an input to this theorem. |
| **Fixed bound and irreducible representations**<br>`theorem forall_irreducible_finrank_le_iff_component_compact_homeomorph (n : ℕ) :`<br>`    (∀ (L : Type u) [NormedAddCommGroup L] [InnerProductSpace ℂ L] [CompleteSpace L],`<br>`      ∀ pi : NonUnitalCStarRepresentation A L, pi.IsIrreducible →` | The natural number $n$ is fixed. For every complete complex Hilbert space $H$ (called $L$ here), every irreducible $*$-representation $\pi : A \to \mathcal{B}(H)$ is tested. Nonunital notation does not require $A$ to lack an identity. |
| **Finite dimension and upper bound**<br>`FiniteDimensional ℂ L ∧ Module.finrank ℂ L ≤ n) ↔` | The two requirements are finite-dimensionality of $H$ and $\dim_{\mathbb{C}} H \leq n$. A finite-rank value alone is not substituted for finite-dimensionality. |
| **Component and applied comparison**<br>`∀ I : NormedSpace.GeometricStructureSpace ℂ A,`<br>`ClosureSpace.HasFiniteDimensionalCompactHomeomorph.{u, u}`<br>`    ((NormedSpace.geometricStructureClosure ℂ A).subspace`<br>`      ((NormedSpace.geometricStructureClosure ℂ A).component I)) (fun m => m ≤ n) :=` | For every geometric point $I \in S(A)$, the ambient closure $c_A$ is restricted to $C_A(I)$. This induced closure is the argument of the compact-model condition. The function of $m$ sets the dimension condition to $m \leq n$ for the same model. |
| **Closure and dimension-condition inputs**<br>`def HasFiniteDimensionalCompactHomeomorph.{uH, uX} {X : Type uX}`<br>`    (c : ClosureSpace X) (P : ℕ → Prop) : Prop :=` | Here $X = C_A(I)$ and the closure argument is $c_{C_A(I)}$; the condition on natural numbers is $P(m) \iff m \leq n$. |
| **Model and inner product**<br>`∃ (H : Type uH) (instN : NormedAddCommGroup H),`<br>`  letI := instN`<br>`  ∃ instI : InnerProductSpace ℂ H,`<br>`    letI := instI` | There exist a model space $K$ (called $H$ in this definition), a normed additive structure and a complex inner product on it. The first local instance makes the normed structure available to the inner-product type; the second makes that same inner product available below. |
| **Completeness and nontriviality**<br>`∃ instC : CompleteSpace H,`<br>`  letI := instC`<br>`  ∃ instT : Nontrivial H,`<br>`    letI := instT` | The same model $K$ is complete and nonzero. Each local instance supplies the chosen structure to the subsequent types; these are not four different spaces. |

| In the source | Mathematical meaning |
| --- | --- |
| **Dimension and one homeomorphism**<br>`FiniteDimensional ℂ H ∧ P (Module.finrank ℂ H) ∧ Nonempty (Homeomorph c (NormedSpace.geometricStructureClosure ℂ (ContinuousLinearMap.compactOperatorStarSubalgebra ℂ H)))` | The same model is finite-dimensional with $\dim_{\mathbb{C}} K \le n$. The final type is inhabited, so there exists a homeomorphism $e : C_A(I) \to S(\mathcal{K}(K))$. The algebra $\mathcal{K}(K)$ consists of compact operators on this same $K$. Both $K$ and $e$ may depend on $I$; $n$ is fixed. |
| **Homeomorphism and its inverse**<br>`structure Homeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where continuous_toFun : c.Continuous d toFun continuous_invFun : d.Continuous c invFun` | The structure extends one bijection $e$; its two continuity fields use that map and its inverse. For all $E \subseteq C_A(I)$ and $F \subseteq S(\mathcal{K}(K))$,<br>$$e(c_{C_A(I)}(E)) \subseteq c_{\mathcal{K}(K)}(e(E)),$$<br>$$e^{-1}(c_{\mathcal{K}(K)}(F)) \subseteq c_{C_A(I)}(e^{-1}(F)).$$<br>These conditions give the closure-preserving equality in the Statement. |

### Exact surrounding source context (separate excerpts)

Exact surrounding source:

```
universe u w
variable {A : Type u} [NonUnitalCStarAlgebra A] [hA : Nontrivial A]
attribute [local instance] ContinuousLinearMap.isClosed_compactOperatorStarSubalgebra
  _root_.CStarAlgebra.spectralOrder _root_.CStarAlgebra.spectralOrderedRing
```

Exact surrounding source:

```
include hA in
```

### Related exact source for the reading guide

**Closure and dimension-condition inputs** · Fixed source

```
def HasFiniteDimensionalCompactHomeomorph.{uH, uX} {X : Type uX}
    (c : ClosureSpace X) (P : ℕ → Prop) : Prop :=
  ∃ (H : Type uH) (instN : NormedAddCommGroup H),
    letI := instN
    ∃ instI : InnerProductSpace ℂ H,
      letI := instI
      ∃ instC : CompleteSpace H,
        letI := instC
        ∃ instT : Nontrivial H,
          letI := instT
          FiniteDimensional ℂ H ∧ P (Module.finrank ℂ H) ∧
            Nonempty (Homeomorph c
              (NormedSpace.geometricStructureClosure ℂ
                (ContinuousLinearMap.compactOperatorStarSubalgebra ℂ H)))
```

**Component and applied comparison** · Fixed source

```
    ∀ I : NormedSpace.GeometricStructureSpace ℂ A,
      ClosureSpace.HasFiniteDimensionalCompactHomeomorph.{u, u}
        ((NormedSpace.geometricStructureClosure ℂ A).subspace
          ((NormedSpace.geometricStructureClosure ℂ A).component I)) (fun m => m ≤ n) :=
```

**Homeomorphism and its inverse** · Fixed source

```
structure Homeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where
  continuous_toFun : c.Continuous d toFun
  continuous_invFun : d.Continuous c invFun
```